Stable reconstruction from scattering data

V. P. Palamodov
Tel Aviv University

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1. Introduction

D - domain in an Euclidean space E^d,

$\mathbf{n}(x)$ - the refraction coefficient ($=$ *inverse velocity*) in D,

The acoustic (Helmholtz) equation:

$$(\Delta + \omega^2 \mathbf{n}^2)u = 0$$

Main problem: *to evaluate* **n** *from boundary measurements of solutions u for a fixed time frequency* ω.

Example: Scattering of a plane wave in a homogeneous background $\mathbf{n}_0$:

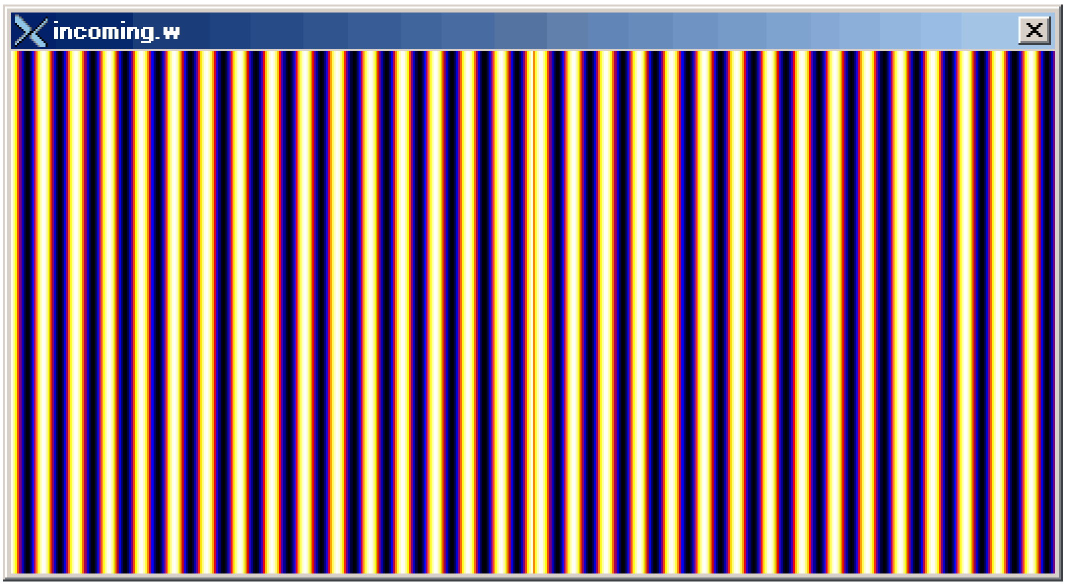
$u = u^{in} + u^{sc}$,

$u^{in} = \exp(2\pi\imath k\langle\theta, x\rangle)$, $k = \omega\mathbf{n}_0$, is an incident planewave,

u^{sc} is the scattered wave on a inhomogeneity $\boldsymbol{\delta} = \mathbf{n} - \mathbf{n}_0$:

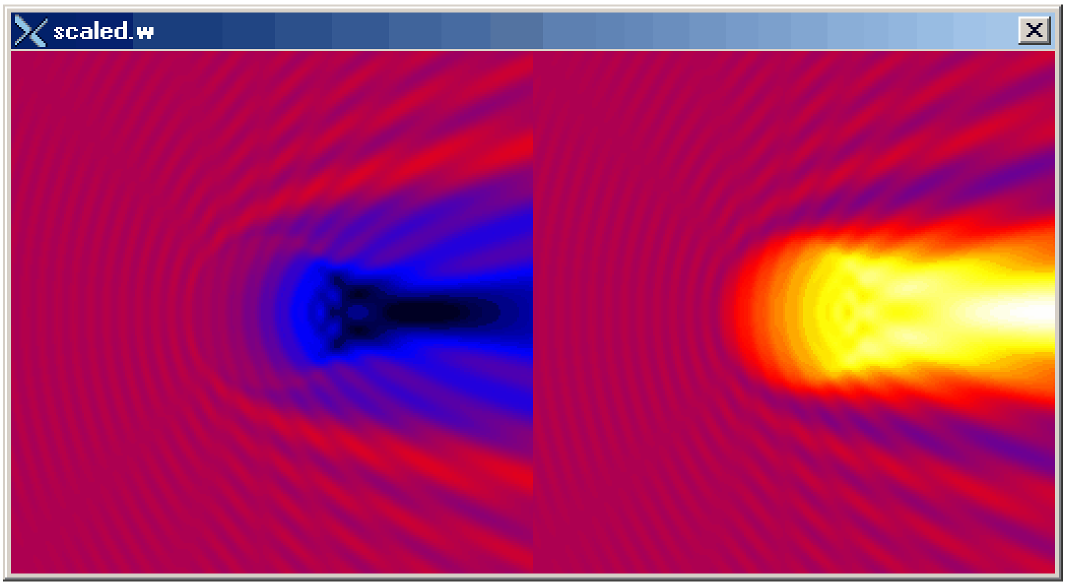
$\mathrm{Re}\, u^{in}$

$\mathrm{Im}\, u^{in}$



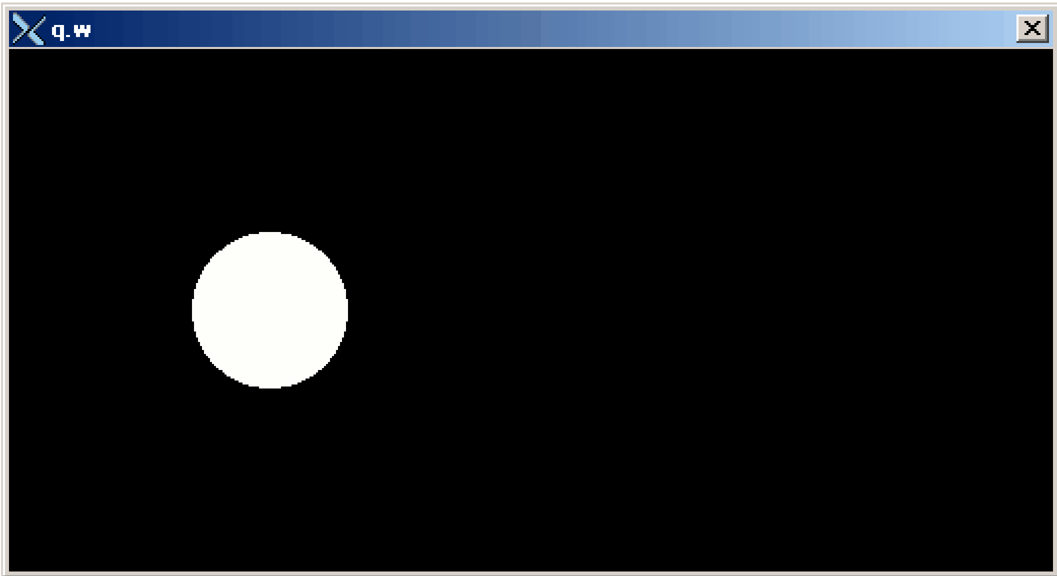
$\mathrm{Re}\, u^{sc}/u^{in}$

$\mathrm{Im}\, u^{sc}/u^{in}$



Reδ

Imδ



Courtesy F.Wuebbeling

2. Analysis of scattering for small perturbation of velocity

Scales of perturbation:

Born perturbation: (validity of Born and Rytov approximations): $\int_\gamma |\delta| ds = o(\frac{1}{\omega})$;

small perturbation:
$\delta = o(\mathbf{n}_0)$.

The last condition is much weaker, since it does not depend on frequency ω.

Constant background velocity:

Set $k = \omega \mathbf{n}_0$ and find a small perturbation $\mathbf{f} = 2\delta\mathbf{n}_0$ from the equation

$$\Delta u + k^2(1 + \mathbf{f})u = 0$$

Take $u(\theta;x) = \exp(\imath k[\langle\theta,x\rangle + w])$.

Neglecting the term $k^2|\nabla w|^2$, we obtain $\Delta w - 2k\langle\theta,\nabla w\rangle = -k\mathbf{f}$.

Fourier transform $\hat{\mathbf{f}}(\xi)$ can be reconstructed in the ball (Ewald ball)

$$\mathsf{B}_{2k} = \left\{\xi \in \mathsf{E}', |\xi| \leq 2k\right\}$$

from the knowledge of the fields $u(\theta;x)$ for all unit vectors θ and all points x in $\partial D \subset \mathsf{E}$.

3. Gabor transform

Phase space $\Phi \doteq \mathsf{E} \times \mathsf{E}'$ of the configuration space E^d.

Liouville volume density $\mathrm{d}\lambda \doteq \mathrm{d}q \wedge \mathrm{d}p$ is canonically defined in Φ.

$q^1,...,q^d,p_1,...,p_d$ are dual coordinates in Φ.

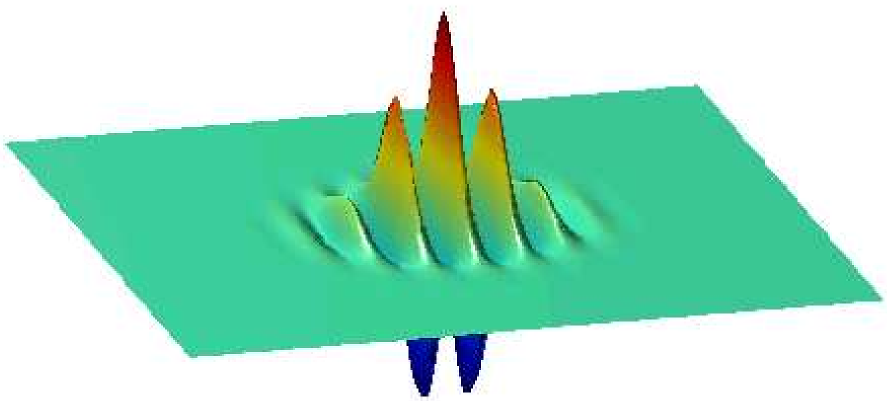
Gabor’s ”*elementary signal*”= *coherent state* in E :

$$\mathbf{e}_\lambda(x) \doteq 2^{d/4}\exp(-\pi(x-q)^2 + 2\pi\imath px),$$
$$\lambda \doteq (q,p) \in \mathsf{E}\times\mathsf{E}'$$

The Gabor function $\mathbf{e}_\lambda$ concentrates ("supported") near q, whereas its Fourier transform $\hat{\mathbf{e}}_\lambda$ concentrates near $p \in \mathsf{E}'$, since

$$\hat{\mathbf{e}}_\lambda = \mathbf{e}_{\hat{\lambda}}, \text{ where } \lambda = (q,p),\ \hat{\lambda} = (p,-q).$$

Portrait of a 2D Gabor function:



The Gabor functions are "asymptotically" orthogonal:

$$|\langle \mathbf{e}_\lambda | \mathbf{e}_\mu \rangle| = \exp\left(-\frac{\pi}{2}|\lambda - \mu|^2\right). \quad (1)$$

For a bounded function **f** the function

$$\mathsf{G}\mathbf{f}(\lambda) = \langle \mathbf{f} | \mathbf{e}_\lambda \rangle, \ \lambda \in \Phi$$

is called *Gabor* transform of **f** (windowed Fourier transform).

The Gabor transform is a isometry $L_2(\mathsf{E}) \to L_2(\Phi)$:

$$\|\mathbf{f}\|^2 = \int_\Phi |\langle \mathbf{f} | \mathbf{e}_\lambda \rangle|^2 \mathrm{d}\lambda.$$

Reconstruction of a function $\mathbf{f} \in L_2$ from its Gabor transform:

$$\mathbf{f}(x) = \int \langle \mathbf{f} | \mathbf{e}_\lambda \rangle \mathbf{e}_\lambda(x) \mathrm{d}\lambda.$$

4. Unstable reconstruction of high frequencies

Space case: $d = 3$

Constant background, *Born perturbation* δ. The scattering amplitude is :

$$a(\theta; x) \approx a_B(\theta; x) = k^2 \int \delta(y) \exp(2\pi \imath k \langle \theta - \tilde{x}, y \rangle) \mathrm{d}y,$$

where a_B is the Born approximation, $\tilde{x} \doteq \frac{x}{|x|}, k = \omega \mathbf{n}_0$ and we use the notation

$\Delta = \sum\left(\frac{\partial}{2\pi\partial x_i}\right)^2$ in the acoustic equation.

Take

$$\boldsymbol{\delta} = \varepsilon\cos(2\pi\langle p,x\rangle)\exp\left(-\pi(x-q)^2\right),$$

for some $\lambda = (q,p), q \in D$ and some small ε (a constant factor is omitted). We have

$$a_B(\theta;x) = \varepsilon k^2[\mathbf{e}_{\mu_1}(\eta) + \mathbf{e}_{\mu_2}(\eta)],$$

where $\eta = k(\theta - \tilde{x}) \in \mathrm{B}_{2k}$ and
$\mu_1 = (p,-q), \mu_2 = (p,q)$.

Both terms $\mathbf{e}_{\mu_1}(\eta)$, $\mathbf{e}_{\mu_2}(\eta)$ are exponentially small as $k \to \infty$, if $|p|/2k \geq 1+\varepsilon$, since $|\eta - p| \geq \varepsilon k$ for arbitrary directions θ, $\tilde{x}$.

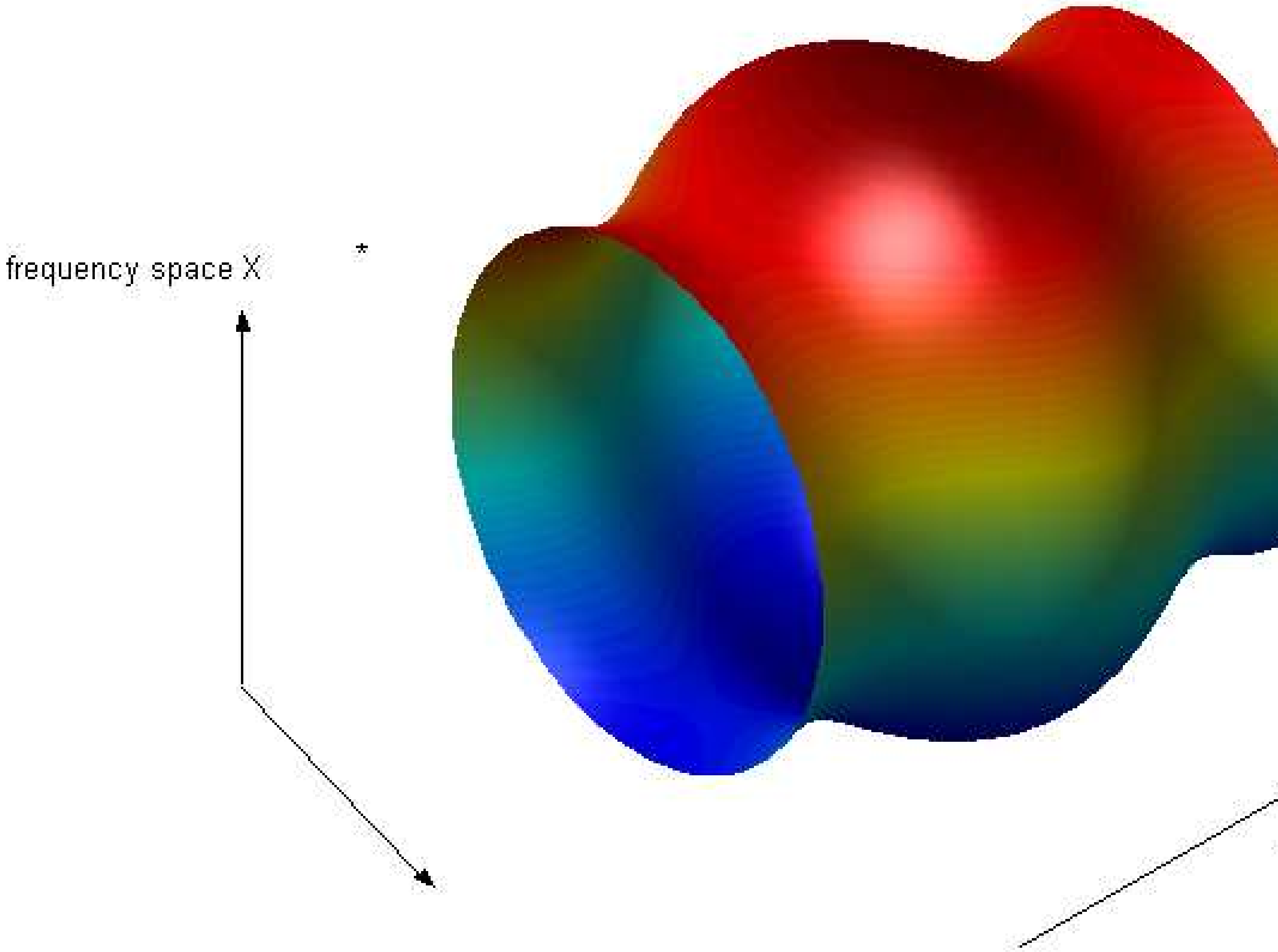
The true amplitude a is also exponentially small.

Conclusion: *Stable reconstruction a perturbation* $\boldsymbol{\delta} = \mathbf{n} - \mathbf{n}_0$ *is* impossible, *if it is concentrated in the complement to the ball* B_R, $R \geq (2+\varepsilon)k$ *for big* k, *since it generates only evanescent waves.*

Theorem. *The same is true for any smooth background* $\mathbf{n}_0, k = \omega\mathbf{n}_0(q)$,

that is stable reconstruction of the Gabor transform of $\boldsymbol{\delta}$ *is impossible in the exterior of the tube-like domain*

$\Omega(\mathbf{n}_0, \omega) \doteq \{|p| \leq 2\omega\mathbf{n}(q)\}$:



Case of constant background: $\Omega(\mathbf{n}_0, \omega) = D \times \mathrm{B}_{2k}$.

Uniqueness results: *L.Faddeev*, *A.Calderon*, *Sylvester-Uhlmann*, *R.Novikov*, *Nachman*, *Eskin-Ralston*, *Sjoestrand*,...

Information on perturbation is extracted from evanescent waves. To amplify this information non-physical solutions like

$$v(x) = \exp(2\pi(\imath\xi x + \eta x)), \;\; \xi,\eta \in \mathsf{C}, \xi^2 - \eta^2 = k^2$$

are coupled with evanescent solutions. The reconstruction is exponentially unstable.

5. Scattering on a line

Case $d = 1$. Write the wave equation in the form

$$\left[\left(\frac{\mathrm{d}}{2\pi \mathrm{d}}\right)^2 + \omega^2 \mathbf{n}^2\right](u^{in} + u^{sc}) = 0,$$

where $k = \omega \mathbf{n}_0$, $\mathbf{n}_0$ be a constant reference coefficient, $\mathbf{n}$ fulfils the only condition: $\mathbf{n}(x) \to \mathbf{n}_0$ as $|x| \to \infty$.

The backscattered wave defines the *reflection* coefficient

$$u^{sc}/u^{in} \to r(k) \text{ as } x \to -\infty.$$

Theorem. *If* $V \doteq \mathrm{Var}_{\mathsf{R}} \log \mathbf{n}/\mathbf{n}_0 < \infty$, *then*

$$\left\| \hat{\delta}(2\omega) - \frac{r(k)}{\imath k} \right\|_{L_2} \leq 2\sinh^2 \frac{V}{4} \left\| \frac{r(k)}{\imath k} \right\|_{L_2},$$

where

$$\hat{\boldsymbol{\delta}}(\boldsymbol{\tau}) = \int \exp(-2\pi\imath\boldsymbol{\tau}\mathbf{t})\boldsymbol{\delta}(x(\mathbf{t}))\mathrm{d}\mathbf{t}, \ \ \boldsymbol{\delta} = \frac{\mathbf{n} - \mathbf{n}_0}{\sqrt{\mathbf{n}\mathbf{n}_0}}$$

and $\mathbf{t} = \mathbf{t}(x) = \int_0^x \mathbf{n}(y)dy$ *is the travel time variable.*

Moreover, for any $\omega > 0$ *the function* $\hat{\boldsymbol{\delta}}(\tau)$ *can be estimated in the interval* $[-2\omega, 2\omega]$ *in terms of the reflection r coefficient in the interval* $[-k, k]$.

N.Grinberg, (1993), *B.Levitan*'s method.

Features of the reconstruction:

- *No smallness assumption on the perturbation* $\boldsymbol{\delta}$.
- *Discontinuous functions* **n** *are allowed.*

Estimate the Gabor transform of $\boldsymbol{\delta}(x)$:

$$\mathrm{G}\boldsymbol{\nu}(\lambda) = \int \exp\left(-\pi(x-q)^2 + 2\pi\imath(px - \tau\mathbf{t}(x))\right)\mathrm{d}x$$

$$\times \int \hat{\boldsymbol{\nu}}(\tau)\mathrm{d}\tau.$$

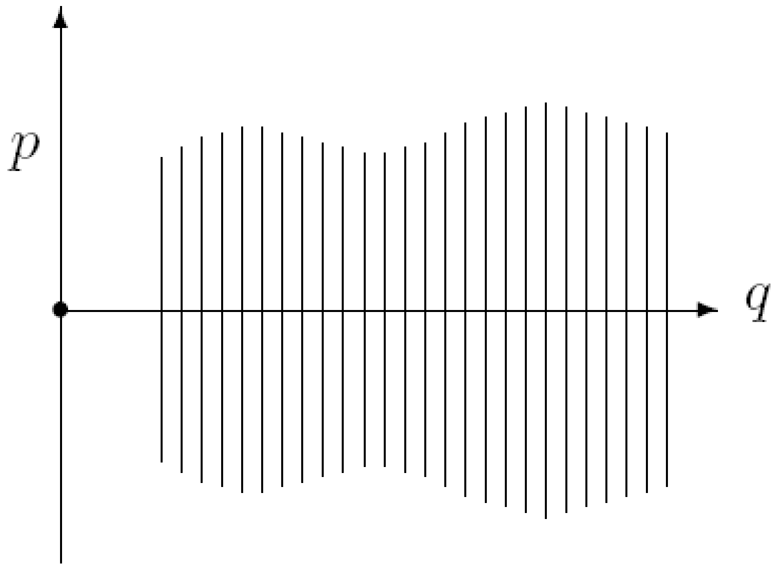
The point $x = q, p = \tau\mathbf{t}'(x) = \tau\mathbf{n}(q)$ gives the main contribution. If $|p| \le 2\omega\mathbf{n}(q),$ then $|\tau| \le 2\omega$ and $\hat{\boldsymbol{\nu}}(\tau)$ is determined from the data of the reflection coefficient r for frequency $k, |k| \le 2\omega\mathbf{n}_0$.

Conclusion. *If we know* $r(k)$ *for*

$|k| \leq \omega \mathbf{n}_0$, the Gabor transform of $\boldsymbol{\delta}$ *is stably reconstructed in the tube-domain*

$$\Omega(\mathbf{n};\omega) \doteq \{(q,p) : |p| \leq 2\omega \mathbf{n}(q)\}.$$

that looks like



Domain of stable reconstruction in 1D

Note that the complementary set to the instability domain as in Sect. 4.

6. Spacial case

Evaluate the difference $\boldsymbol{\delta} = \mathbf{n} - \mathbf{n}_0$ in terms of boundary measurements of solutions u of the equation with unknown coefficient **n**

$$(\Delta + \omega^2 \mathbf{n}^2)u = 0$$

and solutions u_0 of a reference equation

$$(\Delta + \omega^2 \mathbf{n}_0^2)u_0 = 0$$

where we denote $\Delta = \sum\left(\frac{\partial}{2\pi\partial x_i}\right)^2$.

Theorem. *Suppose that* $(\mathbf{n}_0 \mathrm{d}s)^2$ *and* $(\mathbf{n}\mathrm{d}s)^2$ *are smooth non trapping metrics in a bounded domain* $D \subset \mathrm{E}^d, d \geq 3$.

Let $q \in D$ *and* $t_q = \mathrm{d}x/\mathrm{d}\tau$ *be the tangent vector at* q *to a geodesic curve* $\boldsymbol{\gamma}$ *of the metric* **n** (τ is the time parameter), t_q^0 *is the tangent vector to a geodesic curve* $\boldsymbol{\gamma}_0$ *of* $\mathbf{n}_0$. *Then*

$$|\mathrm{G}\boldsymbol{\delta}(\lambda)| \leq C\omega^{-1} \sup_{u,\tilde{u}} \left| \int_{\partial D} (\mathbf{u}\partial_\nu \mathbf{u}_0 - \mathbf{u}_0 \partial_\nu \mathbf{u})\mathrm{d}S \right|$$

$$+ \omega^{-1/2} r(\lambda;\omega), \quad (2)$$

where $\lambda = (q, p = \omega t_q + \omega t_q^0)$ *and* $\mathbf{u}, \mathbf{u}_0$ *are Gauss beam solutions of the corresponding metrics.*

The constant C *and the factor* $r(\lambda;\omega)$ *are uniformly bounded for* $q \in D, \omega \geq \omega_0$ *and* $|t_q \times t_q^0| \geq \varepsilon > 0$.

The statement is also true for the case $d = 2$, *if*

γ and γ_0 have no other common point.

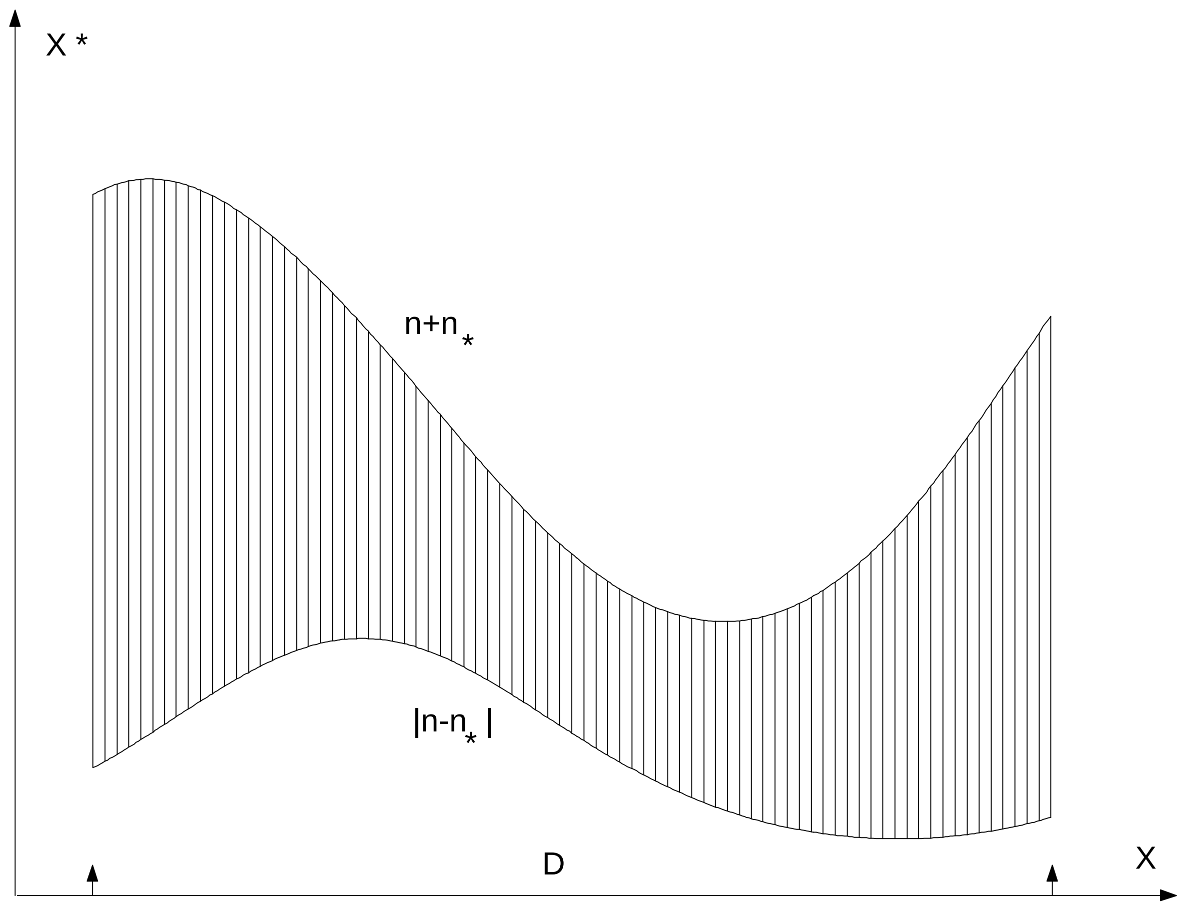
The estimate (2) holds for any point (q, p) in the domain $\Omega(\mathbf{n}, \mathbf{n}_0; \omega)_\varepsilon$ defined by

$$q \in D, \ (1+\varepsilon)\omega|\mathbf{n}(q) - \mathbf{n}_0(q)| \le |p| \le (1-\varepsilon)\omega(\mathbf{n}(q) +$$

with uniformly bounded C and r.

Conclusion. *Stable reconstruction of Gabor transform of the perturbation* $\boldsymbol{\delta}$ *is possible in* $\Omega(\mathbf{n}, \mathbf{n}_0; \omega)_\varepsilon$.

This domain is close to the set $\Omega(\mathbf{n}, \mathbf{n}_0; \omega)_0$:



Perturbative case:

Corollary. *Any small perturbation* $\boldsymbol{\delta} = \mathbf{n} - \mathbf{n}_0$ *can be stably reconstructed in the domain*

$$q \in D,\ \omega\varepsilon \leq |p| \leq \omega(2\mathbf{n}_0(q) - \varepsilon),$$

if $\varepsilon > |\boldsymbol{\delta}|$.

This domain is close to the tube domain $\Omega(\mathbf{n}_0; \boldsymbol{\omega}) \doteq \left\{q \in D,\ |p| \leq 2\omega\mathbf{n}_0(q)\right\}$. See again picture in Sect. 4.

7. Gauss-beam solutions

The case $d = 2$.

Theorem. *If the metric* $(\mathbf{n}ds)^2$ *in* $D \subset \mathsf{E}^2$ *is smooth and non trapping, there exists a frequency* ω_0 *such that for an arbitrary geodesic curve* γ *of the metric and any* $\omega \geq \omega_0$ *there exists a solution* $u_{\gamma,\omega}$ *of the acoustic equation* $(\Delta + \omega^2\mathbf{n}^2)u_{\gamma,\omega} = 0$ *in* D *such that* $u_{\gamma,\omega} = U_\omega + v_\omega$, *where*

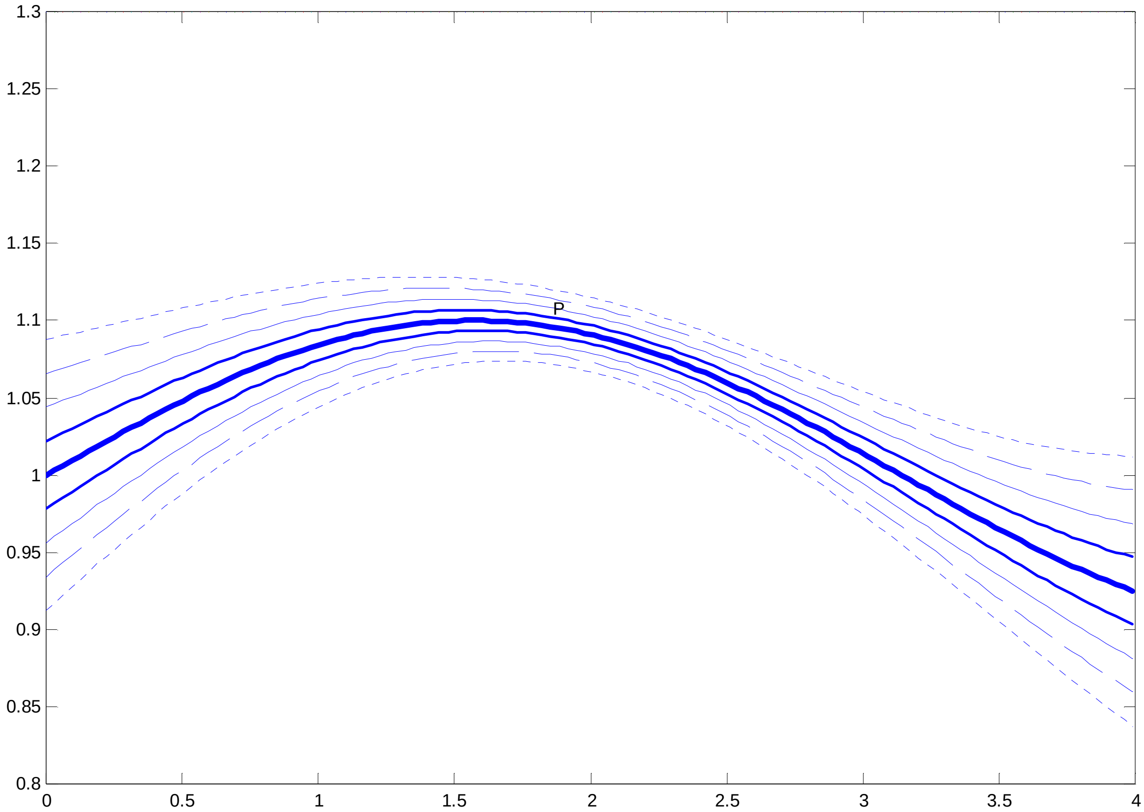
$$U_\omega = (\mathbf{n}\omega j)^{-1/4}\exp(2\pi[\imath\omega\varphi - \omega\psi + \imath\chi]),$$
$$\varphi(s,r) = \int_{s_0}^{s} \mathbf{n}\,d\sigma,$$
$$\psi(s,r) = \frac{\mathbf{n}}{2\mathbf{j}}\left(1 + \imath\left(\log\frac{\mathbf{j}}{\mathbf{n}}\right)'_s\right)r^2,$$
$$\chi(s) = -\int_{s_0}^{s}\frac{d\sigma}{2\mathbf{j}},$$

$\mathbf{j}$ *is the divergence* (*spread function*), s *is the natural coordinate along the curve* $\gamma = \{r = 0\}$, r *is the Euclidean coordinate in the normal direction and* $\int_D |v_\omega|^2 \mathrm{d}x = O(\omega^{-1})$.

The principal term U_ω was found by Leontovich and Fock (1944-6). It has the gaussian profile

$$\exp(-\omega\psi) = \exp\left(-\pi\omega\frac{\mathbf{n}}{\mathbf{j}}r^2\right)$$

in the normal cross-section. A Gauss beam solution $\mathbf{u}_\gamma$ is essentially supported in a $O(L^{1/2})$-nbd of a geodesic $\boldsymbol{\gamma}$, where $L = (\omega\mathbf{n})^{-1}$ is the wave length:

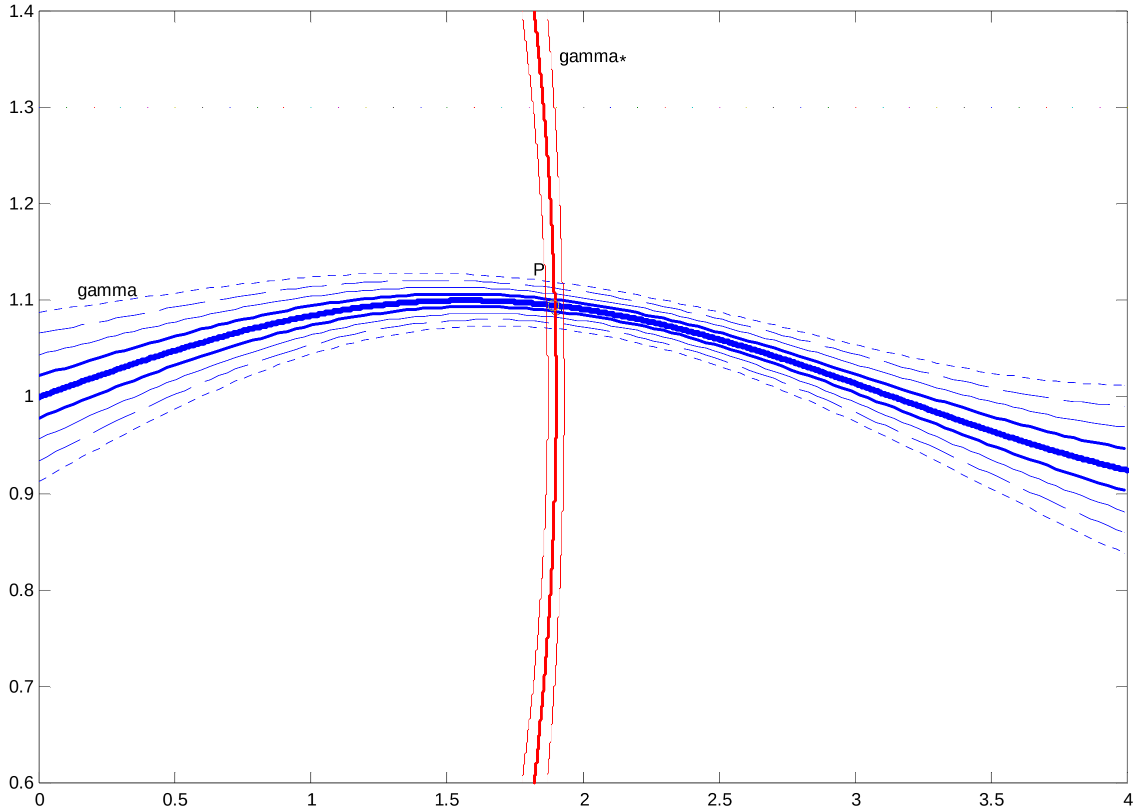


8. Sketch of the proof

Green formula for the Gauss beam solutions

$$\int_{\partial D} (\mathbf{u}\partial_\nu \mathbf{u}_0 - \mathbf{u}_0 \partial_\nu \mathbf{u})\mathrm{d}S = \omega^2 \int_D \boldsymbol{\delta}\,\mathbf{u}\,\mathbf{u}_0\,\mathrm{d}x$$

The product $\mathbf{u}\,\mathbf{u}_0$ is like



which gives evaluation of the integral of the product like

$$\exp(-\pi(r^2 + r_0^2))\exp(2\imath(\varphi + \varphi_0))\boldsymbol{\delta}$$

localized near the point $q \in D$, where φ and φ_0 are respective Eikonal functions. This integral is close to the Gabor transform of $\boldsymbol{\delta}$ at $\lambda = (q, p = t_q + t_q^0)$ in the phase space. ▸

9. Phase space and information

Question:

how much information can we get out from the *data of* G$\delta(\lambda), \lambda \in \Omega(\mathbf{n}; \omega) \subset \mathbf{\Phi}$?

General problem:

how large is dimension of the "space" $\Sigma(\Omega)$ *of functions in* E, *whose Gabor transform is "supported" in a bounded domain* $\Omega \subset \Phi$?

The physical wisdom:

a field (*signal*, *image*) *concentrated in a large bounded domain* $\mathbf{\Omega}$ *of the phase space has about*

$$\mathbf{Vol}\,\mathbf{\Omega} = \int_{\mathbf{\Omega}} \mathbf{d}\lambda$$

independent degrees of freedom (...,*Nyquist*, *Wigner*, *Brillouin*, *Gabor*,...).

A rigorous mathematical theory confirms this guess:

The case $d = 1$, Ω is a box: *Landau*, *Pollak* (1961-2)

The general case: the operator

$$L(\Omega)\mathbf{f} = \int_{\Omega} \langle \mathbf{f} | \mathbf{e}_{\lambda} \rangle \mathbf{e}_{\lambda} \mathrm{d}\lambda$$

in L_2(E) is called *localization* operator for the domain Ω. It is self-adjoint and has discrete spectrum in $(0,1]$.

Definition. *Let* $N(\Omega)$ *be the number of eigenvalues of the operator* $L(\Omega)$ *in the interval*

$[1/2, 1]$ *and* $\Sigma(\Omega)$ *be the subspace in* $L_2(\mathsf{E})$ *spanned by the corresponding eigenfunctions*.

Theorem. [P. 2005] *The number* $N(\Omega)$ *is estimated as follows*

$$|N(\Omega) - \mathrm{Vol}\,(\Omega)| \le C\mathrm{Vol}\,(\partial\Omega_l), (3)$$

where $l = b\log^{d/2}(\mathrm{diam}\,\Omega) + c$ *and* b, c *are some constants and* G_ρ *means l-neighborhood of* G.

In particular, we have

$$\mathrm{Vol}\,\Omega(\mathbf{n}, \mathbf{n}_0; \omega) = b_{\mathsf{E}}\omega^d \int_D \left((\mathbf{n} + \mathbf{n}_0)^d - |\mathbf{n} - \mathbf{n}_0|^d\right)\mathrm{d}x.$$

If the perturbation $\boldsymbol{\delta} = \mathbf{n} - \mathbf{n}_0$ is small, this volume is close to the integral

$$\mathrm{Vol}\,\Omega(\mathbf{n}_0; 2\omega) \sim b_{\mathsf{E}}(2\omega)^d \int_D \mathbf{n}_0^d \mathrm{d}x. \quad (4)$$

where b_{E} means the volume of the unit ball in E.

Corollary. *The number* $N(\omega) \doteq N(\Omega(\mathbf{n}_0; 2\omega))$ *has the same asymptotic expansion* (4).

Proof is not straightforward, since $\mathrm{Vol}\,(\partial\Omega_\rho)$, $\Omega = \Omega(\mathbf{n}_0; 2\omega)$ has the same order $\sim C\omega^d$ of growth as the volume $\mathrm{Vol}\,(\Omega)$ for big ω and constant ρ, since this domain is stretched in

E' directions. The rescaling transformation $q' = \omega^{1/2}q, p' = \omega^{-1/2}p$ in Φ is simplectic and does not change the Liouville volume, whereas it changes the distance functions in E and in E' and the form of a ball in Φ. We now have $\mathrm{Vol}(\partial\Omega_l) = O(\omega^{d/2}\log\omega)$ and (3) yields $N(\omega) \sim \mathrm{Vol}(\Omega)$, Q.E.D.

`Interpretation`: *the quantity* $\mathrm{Vol}\,\Omega(\mathbf{n}_0; 2\boldsymbol{\omega})$ *is equal to the volume of D measured by means of the variable length unity* $\lambda/2 = (2\omega\mathbf{n}_0(x))^{-1}$.

10. **What can we benefit from this knowledge**?

An answer: *Try to find out a reasonable basis in* $\boldsymbol{\Sigma}(\boldsymbol{\Omega}(\mathbf{n_0}; \boldsymbol{\omega}))$.

Take a small $\varepsilon > 0$ and a smooth function $\mathbf{m}$ such that $\mathbf{n}_0 - 3\varepsilon \leq \mathbf{m} \leq \mathbf{n}_0 - 2\varepsilon$ and consider the *Schrödinger operator*

$$\mathsf{S} = -\Delta - 4\omega^2\mathbf{m}^2$$

in D with zero Dirichlet conditions on ∂D. It is self-adjoint and has discrete spectrum.

A ***bound state*** of this operator is an eigenfunction b with non-positive eigenvalue $\lambda = \lambda(b) \leq 0$.

There are only finite number of bound states, since S is bounded from below: $\langle \mathsf{S} g|g\rangle \geq -C\langle g|g\rangle$ for some C.

Theorem. *The set of bound states* $\{b_i\}$ *of* S *is a orthogonal basis in a subspace* $\mathsf{B} \subset \Sigma(\Omega(\mathbf{n}, \omega))$ *such that*

$$\frac{\dim \mathsf{B}}{\dim \Sigma(\Omega(\mathbf{n}_0, \omega))} \geq \left(1 - \frac{3\varepsilon}{\mathbf{n}_0}\right)^d - o(1)$$

as $\omega \to \infty$.

Lemma. *Any bound state of the operator* S *is contained in* $\Sigma(\Omega(\mathbf{n}_0, \omega))$.

Proof. Take a bound state b, $\|b\| = 1$; let $\lambda \leq 0$ be its eigenvalue. The equation

$$-\Delta b = (4\omega^2 \mathbf{m}^2 + \lambda) b$$

yields the estimate

$$\begin{aligned} \|\nabla b\|^2 &= \langle (4\omega^2 \mathbf{m}^2 + \lambda) b | b \rangle \\ &\leq 4\omega^2 \|\mathbf{m} b\|^2, \end{aligned}$$

where $\nabla = \partial / 2\pi \partial x$. For an arbitrary real smooth function $\phi \geq 0$

$$\|\nabla(\phi b)\|^2 = (4\omega^2\mathbf{m}^2 + \lambda)\|\phi b\|^2 + \|\nabla(\phi)b\|^2,$$

which yields

$$\|\nabla(\phi b)\|^2 \le 4\omega^2\|\mathbf{m}\phi b\|^2 + \|\nabla(\phi)b\|^2.$$

The last term is bounded by $M_1(\phi) = \max|\nabla(\phi)|$, which results in

$$\|\nabla(\phi b)\| \le 2\omega\|\mathbf{m}\phi b\| + O(\omega^{-1}). \quad (5)$$

Similar estimate for higher derivatives are

$$\|\nabla^2(\phi b)\| \le (2\omega)^2\|\mathbf{m}\phi b\| + O(\omega), \quad (6)$$

$$\|\nabla^3(\phi b)\| \le (2\omega)^3\|\mathbf{m}\phi b\| + O(\omega^2),$$

where the remainder $O(\omega^2)$ depends on $\max|\nabla\mathbf{m}|$, and so on.

The estimates for the product ϕb looks like bounds for a bandlimited function of the width $2\omega M$, where

$$M = \max\mathbf{m}\phi.$$

Take $\phi_q = \exp\left(-\pi(x-q)^2\right)$ and estimate the integral

$$\mathsf{G}b(\lambda) = \langle b|\mathbf{e}_\lambda\rangle = \int \phi(x)b(x)\exp(-2\pi\imath\langle x,p\rangle)\mathrm{d}x$$

By partial integration:

$$\mathrm{G}b(\lambda) = \frac{1}{\imath|p|^2}\int\langle p,\nabla\rangle(\phi b)\exp(-2\pi\imath\langle x,p\rangle)\mathrm{d}x,$$

since
$\langle p,\nabla\rangle\exp(-2\pi\imath\langle x,p\rangle) = -\imath|p|^2\exp(-2\pi\imath\langle x,p\rangle)$.
By the Schwarz inequality and by (5)

$$|\mathrm{G}b(\lambda)| \leq \frac{|D|^{1/2}}{|p|}\|\nabla(\phi b)\|$$

$$\leq |D|^{1/2}\frac{2\omega M(q)}{|p|}(1+O(\omega^{-2})),$$

where $M(q) = \max \mathbf{e}_{(q,0)}\mathbf{m}$ and $|D| = \int_D \mathrm{d}x$.
Integrate by parts again and apply (6):

$$|\mathrm{G}b(\lambda)| \leq \frac{|D|^{1/2}}{|p|^2}\|\nabla^2(\phi b)\|$$

$$\leq |D|^{1/2}\left(\frac{2\omega M(q)}{|p|}\right)^2(1+O(\omega^{-1}))$$

and so on. This implies that $|\mathrm{G}b(\lambda)|$ is small for $|p| > 2\omega M(q)$ and decrease as $|p| \to \infty$. We can fulfil the condition $M < \mathbf{n}_0 - \varepsilon$, if we replace ϕ_q by the function
$\phi_{\sigma,q} = \exp\left(-\pi\sigma(x-q)^2\right)$, which is better localized at the point q, if $\sigma > 1$. This is equivalent to rescaling Euclidean product in E, which causes dual rescaling in E$'$ and does not change the Liouville volume.

By Sect. 3 we can write

$$b = b_1 + b_2 \doteq \int_{\Omega'} \mathrm{G}b(\lambda)\mathbf{e}_\lambda \mathrm{d}\lambda + \int_{\Phi\backslash\Omega'} \mathrm{G}b(\lambda)\mathbf{e}_\lambda \mathrm{d}\lambda$$

where $\Omega' = \Omega(\mathbf{n}_0 - \varepsilon; \omega)$. The norm $\|b_2\|$ is small, since of the aforesaid. Apply the localization operator of the complementary domain $\Psi = \Phi\backslash\Omega,\ \Omega = \Omega(\mathbf{n}_0; \omega)$:

$$L(\Psi)b = L(\Psi)b_1 + L(\Psi)b_2$$

The second term has small norm since $\|L(\Psi)\| \leq 1$. By (1) the first term is also small, since the domains Ω' and Ψ are disjoint and the distance is positive and growing with ω. Therefore $\|L(\Psi)b\| \leq 1/2$ and

$$\begin{aligned} \|L(\Omega)b\| &= \|b - L(\Psi)b\| \\ &\geq \|b\| - \|L(\Psi)b\| \geq 1/2, \end{aligned}$$

which implies $b \in \Sigma(\Omega),$ Q.E.D.

Proof of Theorem. Let B be the linear span of bound states; the bound states form an orthogonal basis. By the Lemma B $\subset \Sigma(\Omega(\mathbf{n}_0, \omega))$. Compare the dimensions. By Corollary of Sect. 9 we have

$$\dim \Sigma(\Omega(\mathbf{n}_0, \omega)) = N(\omega) \sim b_{\mathsf{E}}(2\omega)^d \int \mathbf{n}_0^d \mathrm{d}x.$$

since the volume of According to the spectral theory the dimension of B has the asymptotic evaluation

$$\text{dim B} = (1 + o(1))b_{\mathsf{E}}(2\omega)^d \int \mathbf{m}^d \mathrm{d}x$$

as $\omega \to \infty$. (Birman-Solomyak, 1970) and the theorem follows. ▸

Conclusion. *For any smooth* $\mathbf{m} < \mathbf{n}_0$ *a stable reconstruction of the refraction coefficient* $\mathbf{n}$ *is possible in the form*

$$\boldsymbol{\delta} \equiv \mathbf{n} - \mathbf{n}_0 = \sum_1^{\beta} c_i \mathbf{b}_i, \;\; \beta \doteq \text{dim B}.$$

where $\mathbf{b}_1, ..., \mathbf{b}_\beta$ *are all bound states for the operator* $\mathsf{S} = -\Delta - 4\omega^2\mathbf{m}^2$ *and the coefficients* $c_1, ..., c_\beta$ *are uniquely defined.*

The number β *is the maximal dimension up to a factor* $(1 - \varepsilon)$ *of manifolds, where a stable reconstruction of* $\boldsymbol{\delta}$ *is possible.*

Hint: *If you have already an approximation* $\tilde{\boldsymbol{\delta}}$ *to* $\boldsymbol{\delta}$, *filter it by projecting to* B !

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