# Least Squares Inversion Revisited

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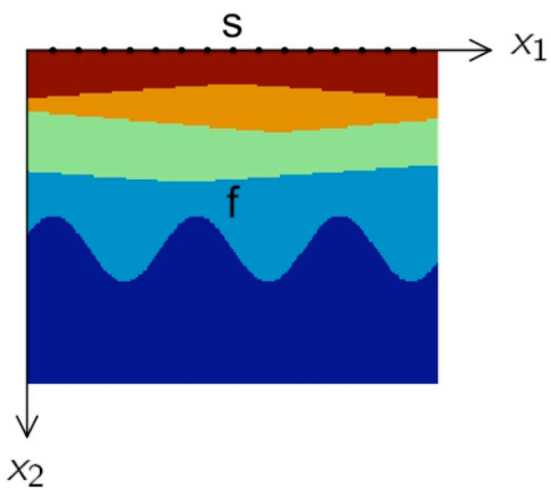
# The model problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u,\ x_2 > 0,\ 0 \le t \le T,\ u = 0\ (t < 0)$$

$$\frac{\partial u}{\partial x_2}(x_1, 0) = q(t) p(x_1 - s), \qquad \mathrm{c}^2 = \frac{c_0^2}{1+f}$$

$$g_s(x_1, t) = u(x_1, 0, t) = (R_s(f))(x_1, t) \text{ seismogram for source } s$$



# Kaczmarz' method

Solve $R_s(f) = g_s$ for all sources $s$.

Update:

$$f \longleftarrow f - \alpha (R_s'(f))^* (R_s(f) - g_s)$$

# What can we expect from reflection data?

$\hat{f}(\rho+\sigma,-a(\rho)-a(\sigma))$ determined

for $|\rho|,|\sigma| \le k$, $a(\rho)=\sqrt{k^2-\rho^2}$.

$\hat{f}$ determined

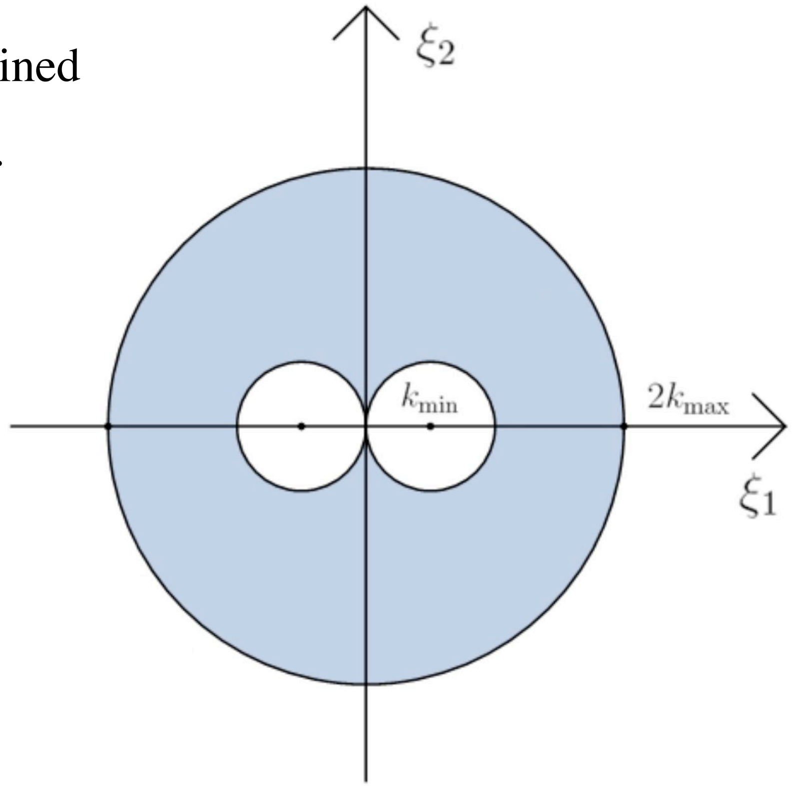
in blue domain.

$k=\omega/c_0$, $\omega \in [\omega_{min},\omega_{max}]$

$\omega_{min},\omega_{max}$ minimal

and maximal frequency

of source wavelet $q$



# Numerical experiments

$c_0 = 2km/\sec$

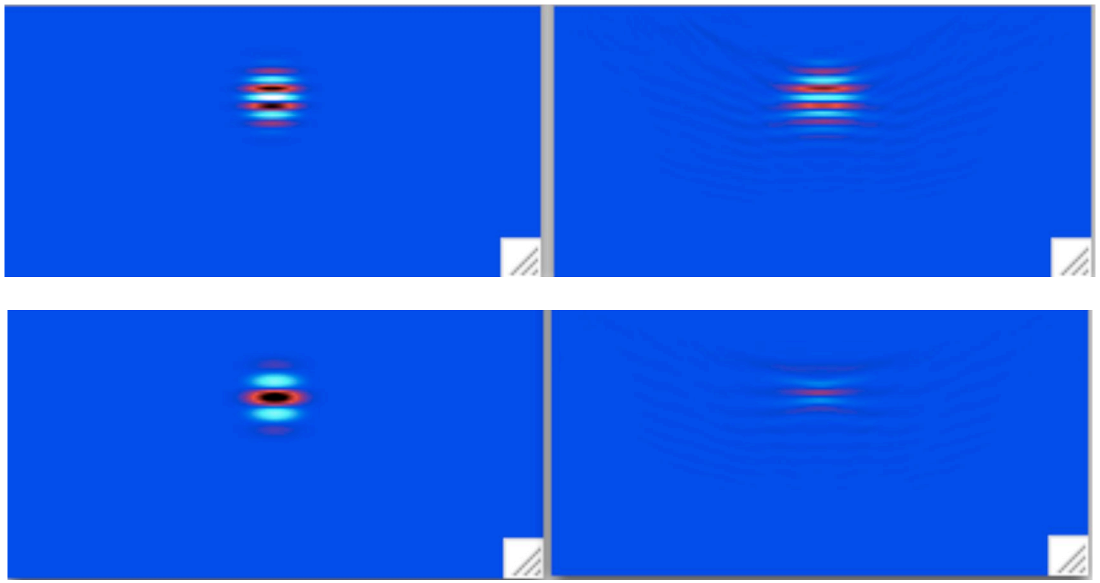
lowest frequency $f = 5Hz$

depth $6km$, $T = 6\sec$

original | reconstruction

$$f(x_1, x_2) = \cos(2\pi f x_2 / c_0)$$

$$f(x_1, x_2) = \cos(2\pi \frac{f}{2} x_2 / c_0)$$

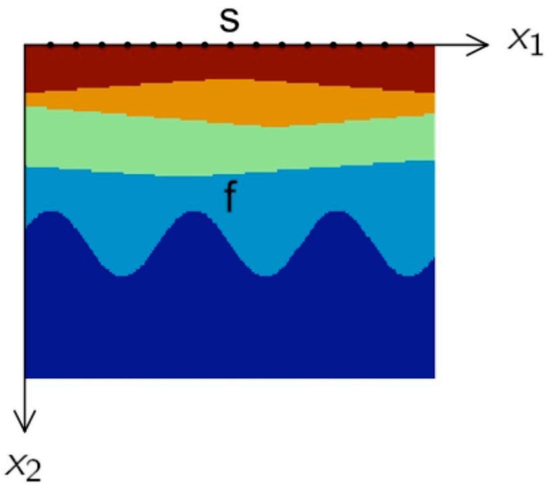


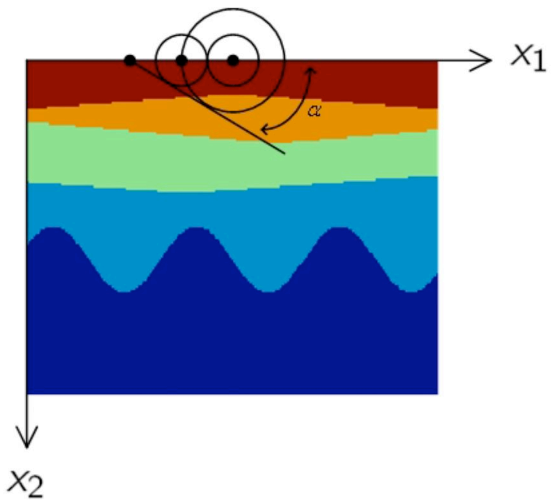
Kaczmarz does exactly what you can possibly expect!

# Plane wave stacking

$$\frac{\partial u}{\partial x_2} = q(t)p(x-s)$$

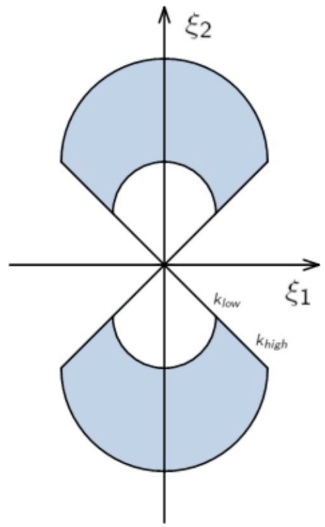
$$\frac{\partial u}{\partial x_2} = \sum_s q(t - \sin(\alpha)s/c_0)p(x-s)$$



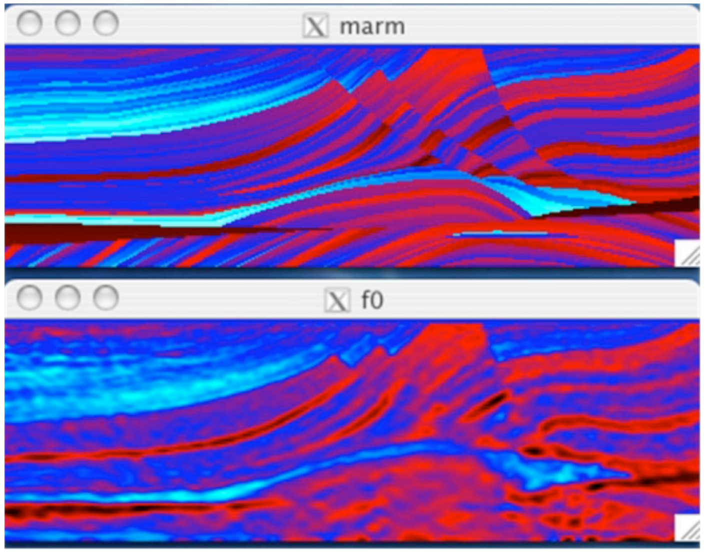

$$(R_\alpha(f))(x_1,t) = \sum_s (R_s(f))(x_1, t - \sin(\alpha)s/c_0)$$

# What can we expect from plane waves?



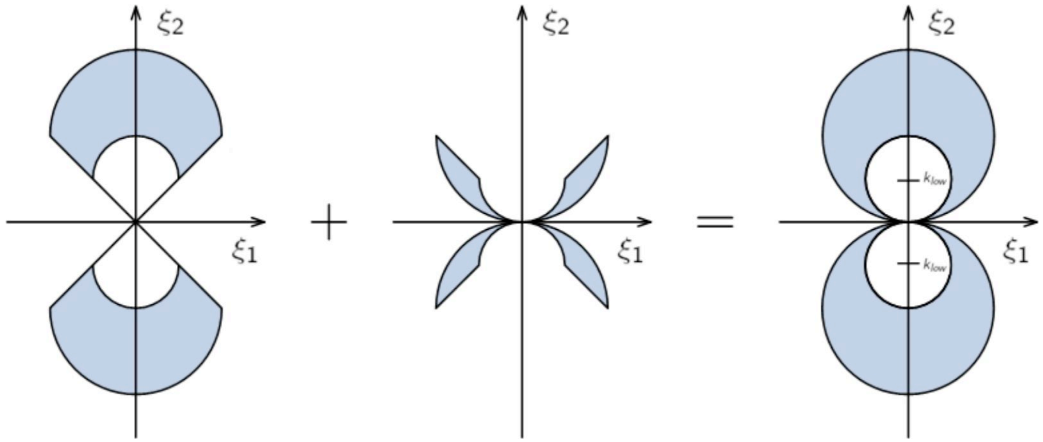

If the source wavelet q has frequencies down to 0, f can be recovered from 2 plane waves making an angle of 90°.

# Marmousi with Kaczmarz and plane wave stacking

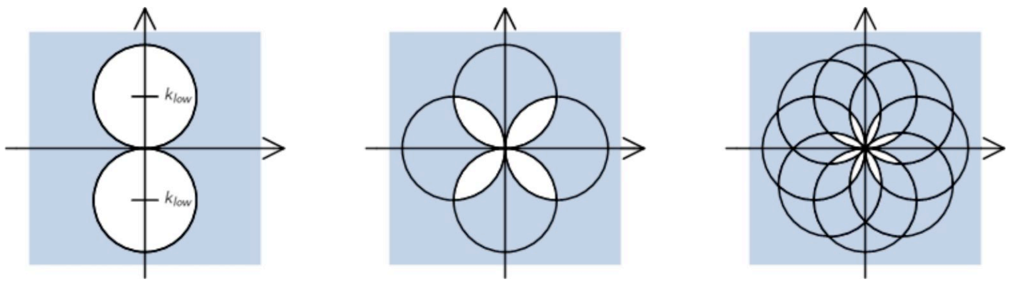



Works only for delta-like source wavelets q!

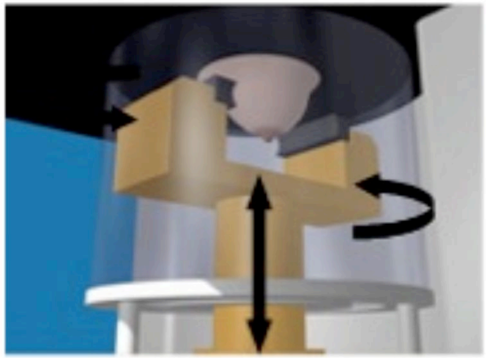
# Combining reflection with transmission

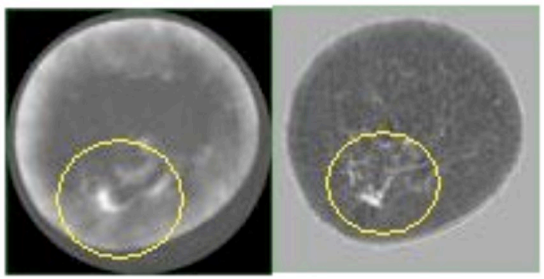



# Combining 1, 2, 4 waves

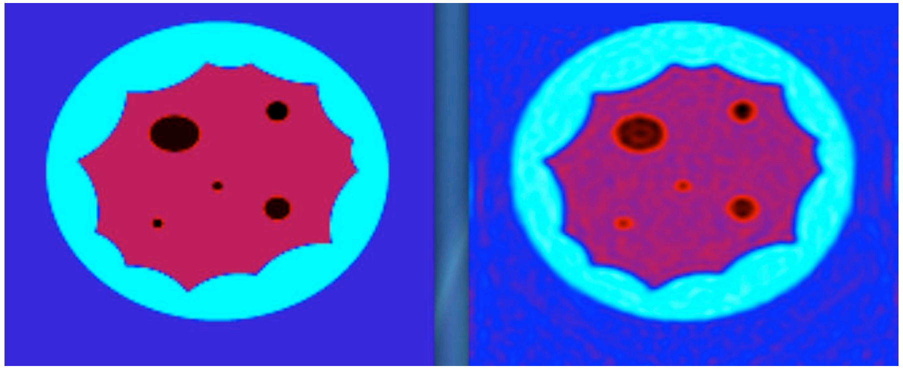


# Ultrasound mamography





# Kaczmarz with plane wave stacking in medical imaging



original reconstruction

Computing time <1 minute on a 3Ghz double processor PC

# Kaczmarz in frequency domain

$$\Delta U + k^2(1+f)U = 0,\ x_2 < 0,\ k = \omega / c_0$$

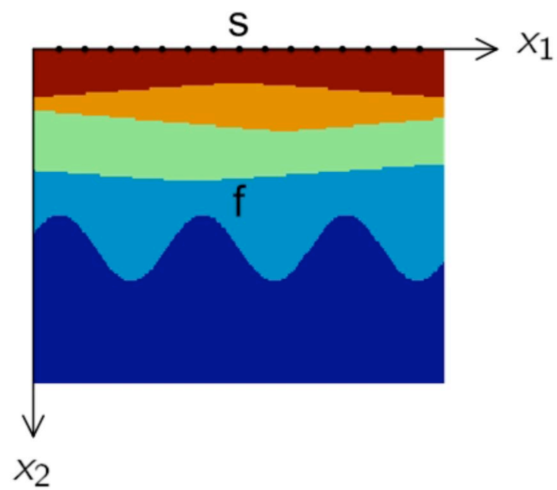
$$\frac{\partial U}{\partial x_2}(x_1,0) = \hat{q}(\omega)p(x-s)$$

$$U(x_1,0) = \hat{g}_s(x_1,\omega)$$

$U$ radiation condition for $|\mathrm{x}| \to \infty$:

$$(\hat{R}_s(f))(x) = \frac{\partial U}{\partial r} - ikU = 0 \text{ on } |x| = r,\ r \text{ large.}$$

Solve $\hat{R}_s(f) = 0$ for all sources.



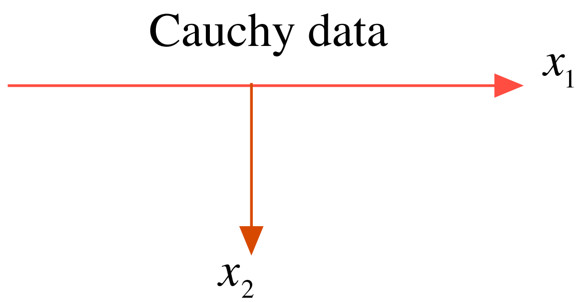

One-way or two-way
wave equation?

# Initial Value Problem for the Helmholtz Equation

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + k^2 u = 0 \qquad U(x_1,0) = U_0(x_1),\ \frac{\partial U}{\partial x_2}(x_1,0) = U_1(x_1)$$

Fourier transform with respect to $x_1$:

$$\hat{U}(\xi_1,x_2) = (2\pi)^{-1/2} \int \exp(-i x_1 \xi_1) U(x_1,x_2)\, dx_1$$




Ordinary differential equation in $x_2$:

$$\frac{d^2\hat{U}(\xi_1,x_2)}{dx_2^2} + (k^2 - \xi_1^2)\hat{U}(\xi_1,x_2) = 0$$

Solution:

$$\hat{U}(\xi_1,x_2) = \hat{U}_0(\xi_1)\cos(\kappa(\xi_1)x_2) + \frac{\hat{U}_1(\xi_1)}{\kappa(\xi_1)}\sin(\kappa(\xi_1)x_2),\ \kappa(\xi_1) = \sqrt{k^2 - \xi_1^2}$$

Stable as long as $\xi_1^2 \le k^2$

# Stability Estimates for the Cauchy Problem of the Inhomogeneous Helmholtz Equation

$$\Delta U + k^2(1+f)U = r, \qquad x_2 > 0, \qquad f \geq 1 + m_1, \;\; m_1 > -1$$

$$U(x_1, 0) = 0, \qquad \frac{\partial u(x_1, 0)}{\partial x_2} = 0,$$

$U_\kappa =$ low pass filtered (in $x'$) version with cut-off $\kappa$ of $U$

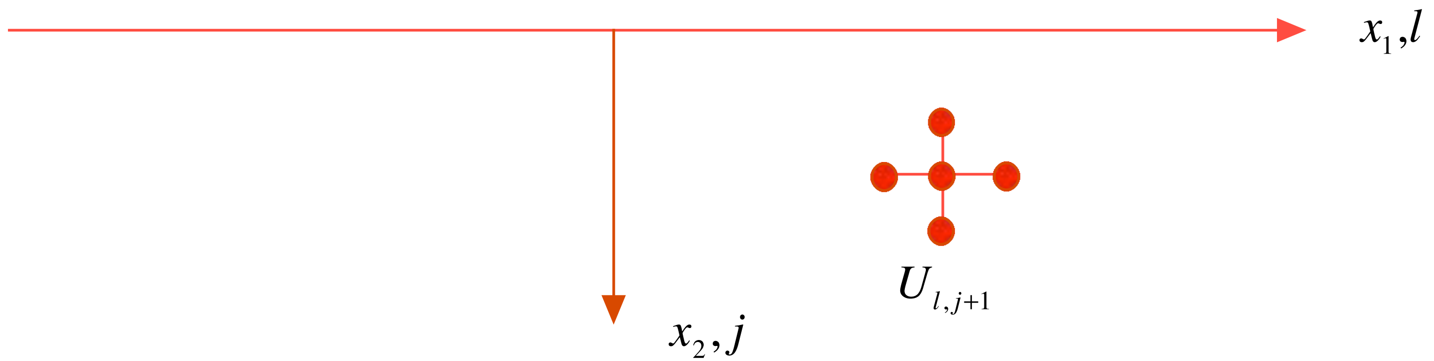
$$\left\| U_{\kappa\vartheta}(x_1, x_2) \right\|_{L^2(R^1)} \leq \frac{c(x_2)}{\kappa\vartheta} \left\| r \right\|_{L^2(R^1 \times [0, x_1])}, \;\; \kappa = k(1 + m_1), \;\; 0 < \vartheta < 1$$

# Stable marching for the Helmholtz equatior

Compute a preliminary value $U^*_{l,j+1}$ from

$$-4U_{l,j} + U^*_{l,j+1} + U_{l,j-1} + U_{l+1,j} + U_{l-1,j} + h^2k^2(1+f_{l,j}) = 0$$

Compute $U_{l,j+1}$ by low pass filtering of $U^*_{l,j+1}$with respect to $l$

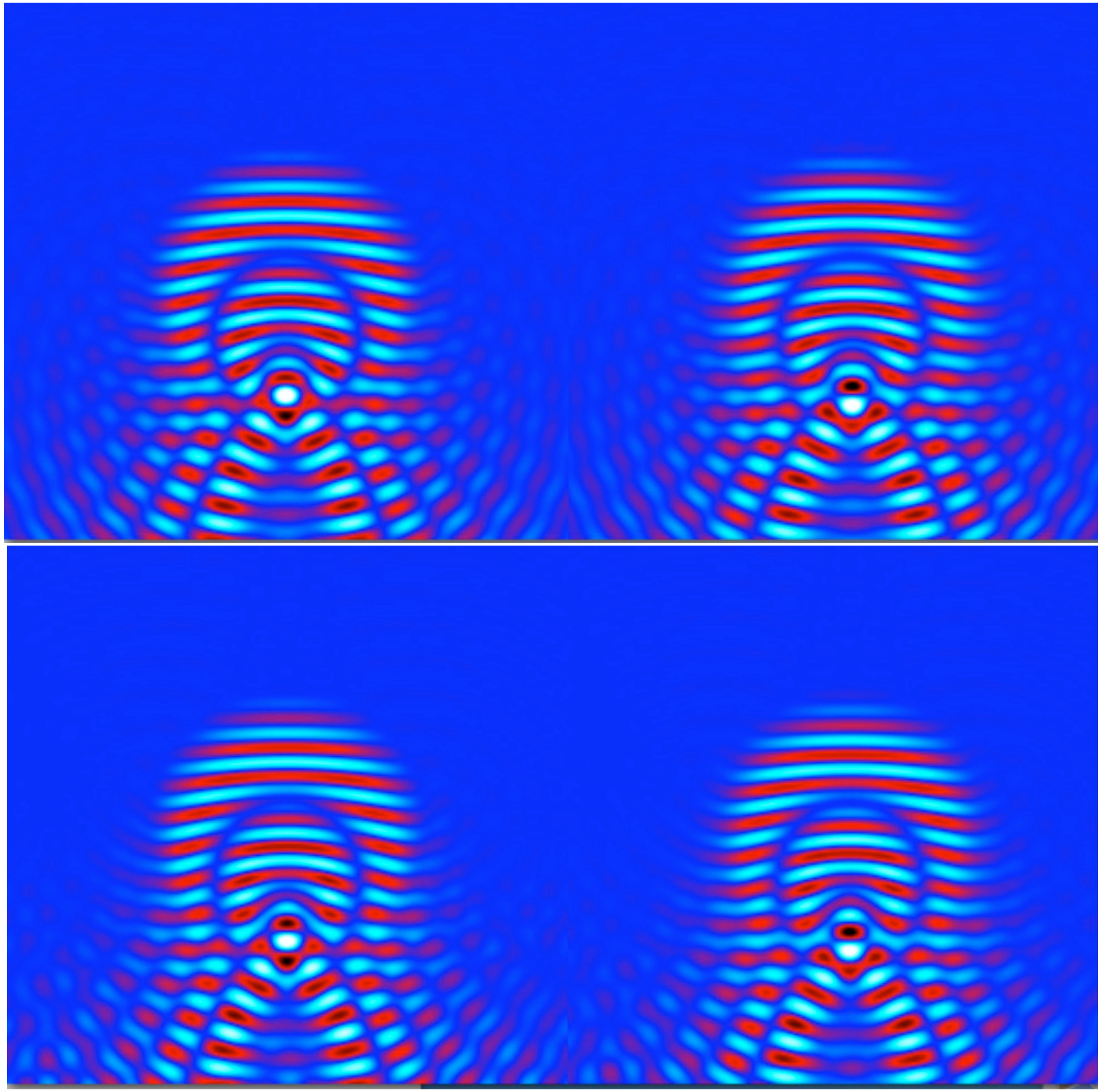



$n^2 \log(n)$ flops on $\mathrm{n} \times \mathrm{n}$ grid!

Exact (finite difference time domain, followed by Fourier transform)

# LUNEBERG LENSE

Initial value technique

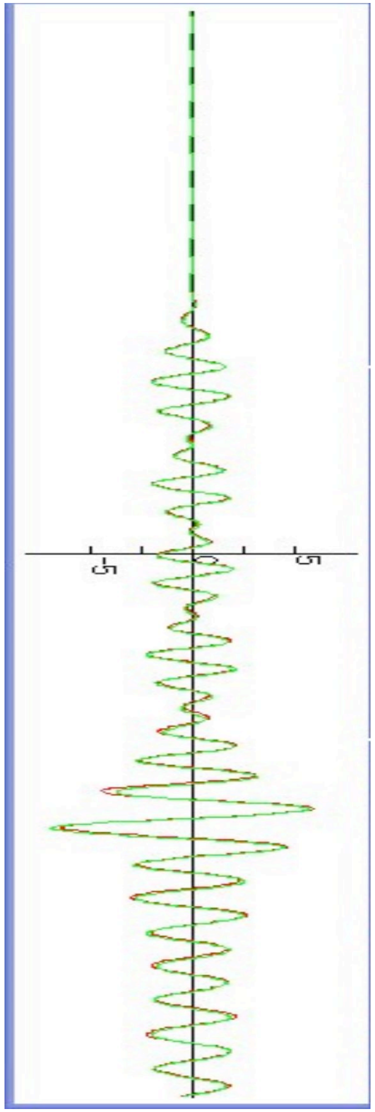


Vertical cross section

through real part

green: exact

red: initial value technique




focal point

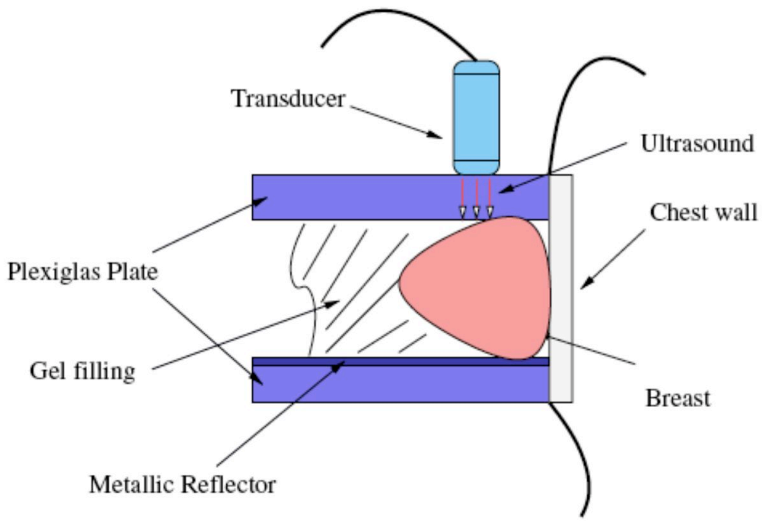
# What can we do for „real“ source wavelets q?

Make use of reflectors!

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u,\ x_2 > 0,$$

$$\frac{\partial u}{\partial x_2} = q(t) p(x - s),\ x_2 = 0$$

$$\frac{\partial \mathrm{u}}{\partial \mathrm{x}_2} = 0,\ x_2 = D$$




CARI

# Theory of CARI

One can show that

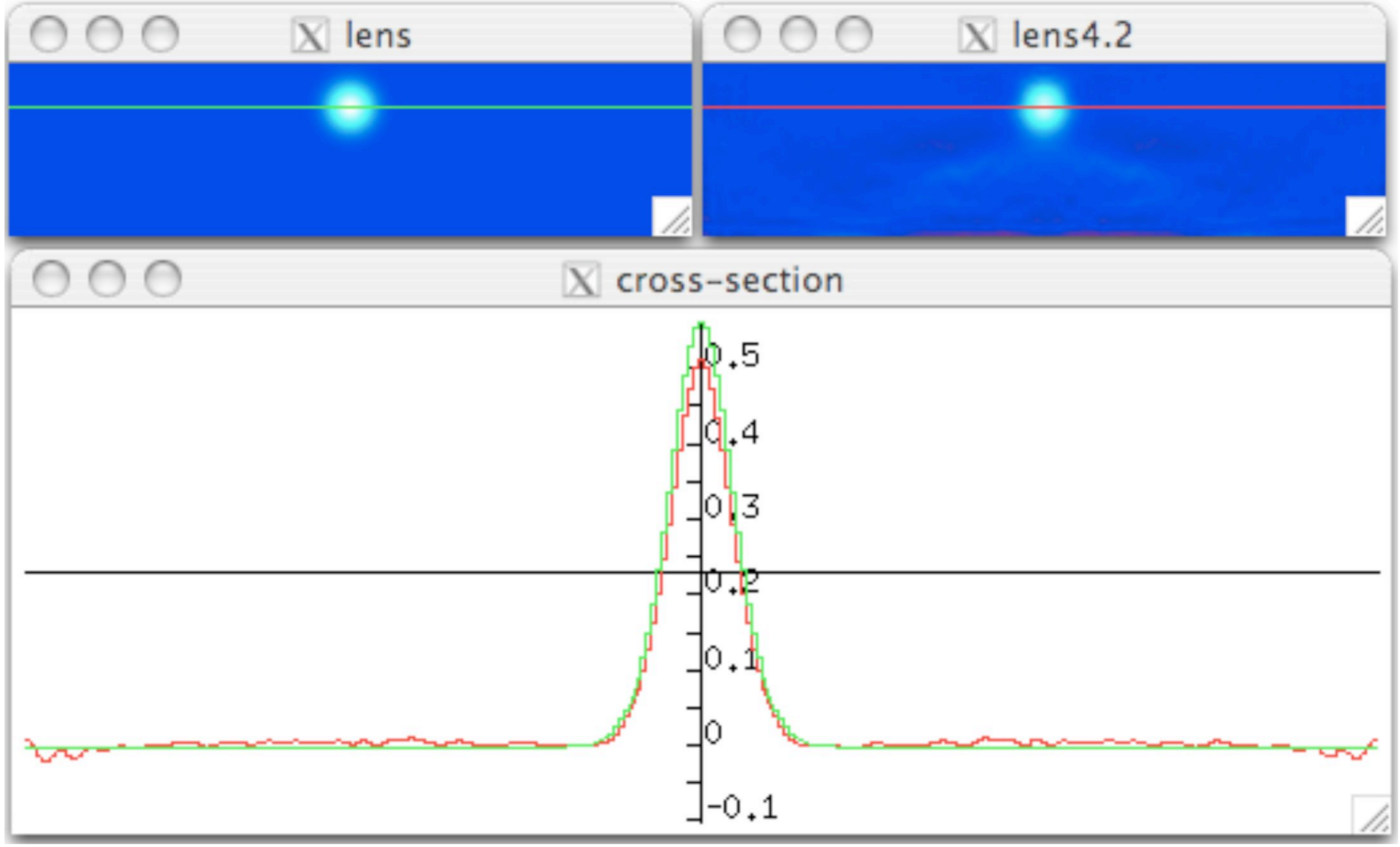
$$XY\hat{f}(\rho+\sigma,a(\rho)+a(\sigma))+\frac{X}{Y}\hat{f}(\rho+\sigma,a(\rho)-a(\sigma))+$$
$$\frac{Y}{X}\hat{f}(\rho+\sigma,-a(\rho)+a(\sigma))+\frac{1}{XY}\hat{f}(\rho+\sigma,-a(\rho)-a(\sigma))$$

is uniquely determined for $|\rho|,\ |\sigma|\leq k$.

$$a(\rho)=\sqrt{k^2-\rho^2},\ \ X=\exp(ia(\rho)D),\ \ Y=\exp(ia(\sigma)D)$$

Low frequency terms are involved, too!

# Numerical experiments with CARI




$$c(x_1,x_2)=c_0-a\exp(-(x_1^2+(x_2-0.5)^2)/0.09),\ a=0.4,\ c_0=2km/\sec$$

# Unknown reflectors

$$f(x) = f_0(x) + f_1(x)$$

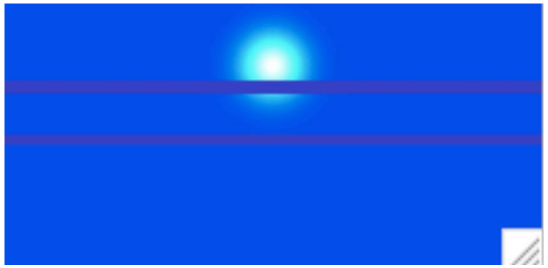
smooth background (unknown) → $f_0(x)$

reflectors (unknown) → $f_1(x)$

We do not distinguish between background and reflectors!

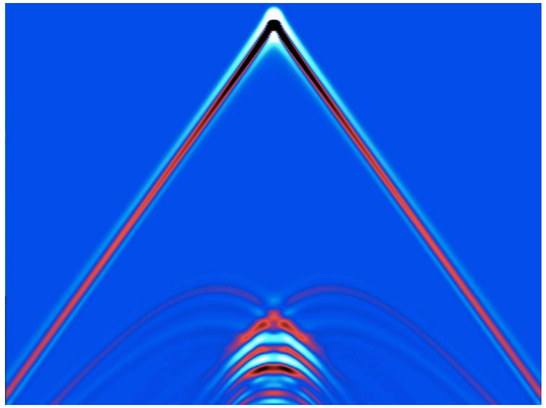
# Numerical experiment

$$c(x_1,x_2)=\begin{cases} c_0 - a\exp(-(x_1^2+(x_2-0.5)^2)/0.09),\ a=0.8km/\sec,\ c_0=2km/\sec \\ 2.5km/\sec \text{ in two horizontal strips}\end{cases}$$



$f$

$$c^2=\frac{c_0^2}{1+f}$$



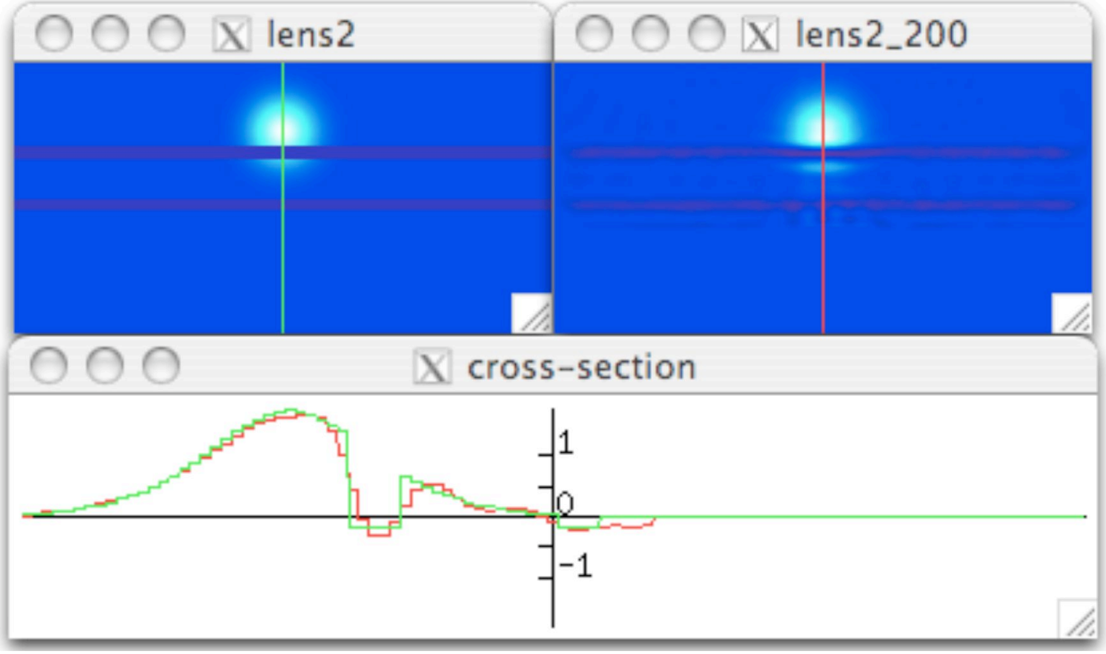
common shot gather

Ricker wavelet

peak frequency at 18 Hz

# Numerical experiment

original




reconstruction

Data band-pass filtered to 2.5-5 Hz

Why do we need low frequencies?

# (Highly) necessary condition for convergence

Initial approximation $f_0$, corresponding field $U_0$: $\Delta U_0 + k^2(1+f_0)U_0 = 0$

True $f$, true field $U$: $\Delta U + k^2(1+f_0)U = -k^2(f-f_0)U$

Linearization: $\Delta U + k^2(1+f_0)U = -k^2(f-f_0)U_0$

Valid only if at least $|\,\mathrm{phase}(U) - \mathrm{phase}(U_0)\,| < \pi$

WKB: $U \approx U_0 \exp\left\{ \frac{\mathrm{ik}}{2} \int (f-f_0)\,ds \right\}$

$$\left| \int (f-f_0)\,ds \right| < \frac{2\pi}{k} = \lambda$$

# Necessary conditions for lens

$$\int f(x)ds = 0.74 \text{ km}$$

At 5 Hz: $\lambda = 0.4$ km no convergence

At 2.5 Hz: $\lambda = 0.8$ km convergence

# Conclusions

Least squares (Kaczmarz) achieves what you can reasonably expect

Start with sufficiently low frequencies

Use frequency domain initial value techniques