Wave Field Decomposition, Seismic Interferometry, and Time Reversal Mirrors

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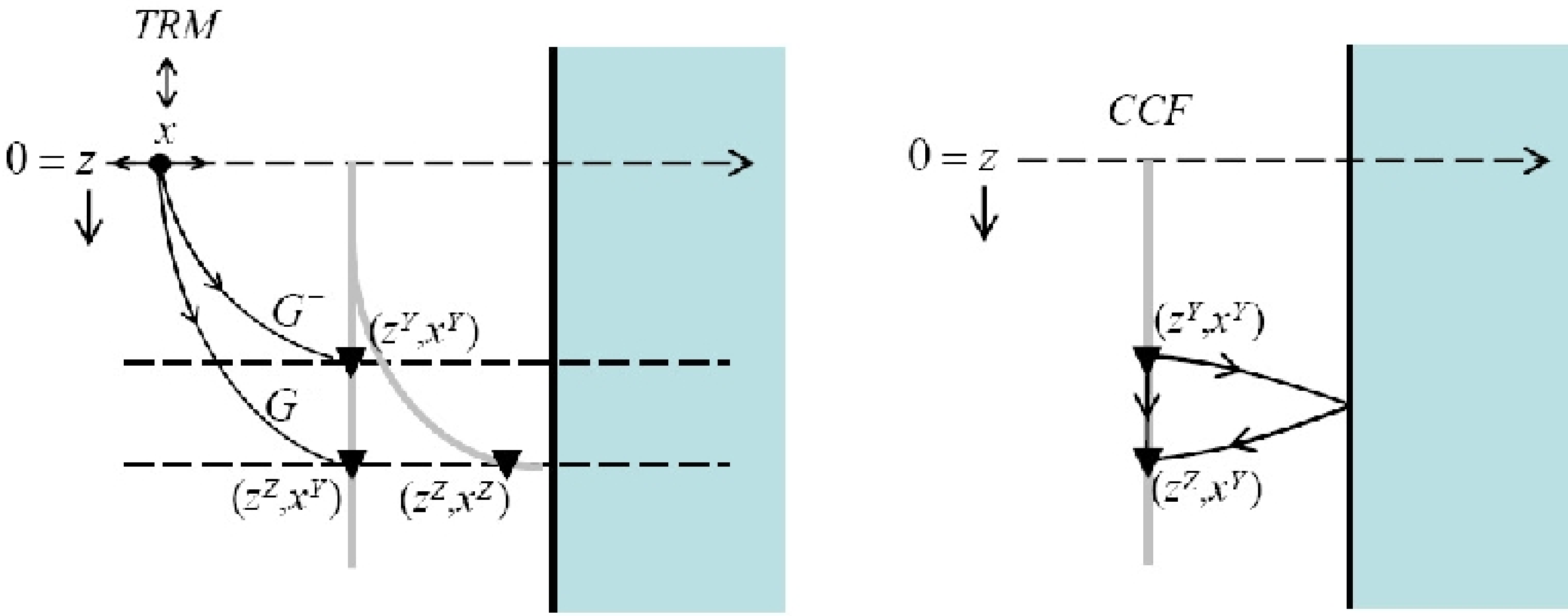
Seismic Interferometry



CCF: 3-d VSP, sources at surface, receivers in vertical well(s), apply s/r reciprocity, CCF via integration over the sources

One-Sided TRM: propagate from source plane to mirror plane, record, reverse recording, propagate to receiver plane, record

Seismic Interferometry

Cross Correlation Function (CCF)

$$CCF(\omega) = \int_{\mathbb{R}^2} d\underline{x} G^{-}\left(\omega; z^Y, \underline{x}^Y; z, \underline{x}\right)\left[G^{+}\left(\omega; z, \underline{x}; z^Z, \underline{x}^Z\right)\right]^* \quad \text{(FD)}$$

$$CCF(t) = \left(\frac{1}{2\pi}\right)\int_{\mathbb{R}} d\omega \exp(-i\omega t)$$

$$\bullet \int_{\mathbb{R}^2} d\underline{x} G^{-}\left(\omega; z^Y, \underline{x}^Y; z, \underline{x}\right)\left[G^{+}\left(\omega; z, \underline{x}; z^Z, \underline{x}^Z\right)\right]^* \quad \text{(TD)}$$

One-Sided Time Reversal Mirror Experiment (TRM)

$$TRM(t) = H(t)\int_0^{t^+} ds \int_{\mathbb{R}^2} d\underline{x}$$

$$\bullet G^{-}\left(t-s; z^Y, \underline{x}^Y; z, \underline{x}\right) G^{+}\left(T-s; z, \underline{x}; z^Z, \underline{x}^Z\right)$$

Wave Field Modeling

Scalar Helmholtz Equation

$$\left(\partial_z^2 + \underbrace{\nabla_{\underline{x}}^2 + \bar{k}^2 K^2(z,\underline{x})}_{\bar{k}^2 \mathbf{B}^2}\right)\phi(z,\underline{x}) = -\delta(\underline{x}-\underline{x}_s)\delta(z-z_s)$$

Wave Field Decomposition

$$\begin{pmatrix} \left(i/\bar{k}\right)\partial_z + \mathbf{B} & 0 \\ 0 & \left(i/\bar{k}\right)\partial_z - \mathbf{B} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} =$$

$$\begin{pmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{R} & \mathbf{T} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \left(1/2\bar{k}^2\right) \mathbf{B}^{-1} \delta(\underline{x}-\underline{x}_s)\delta(z-z_s)$$

Wave Field Modeling

where

$$\phi^{\pm}(z,\underline{x}) = \frac{1}{2}\left[\phi(z,\underline{x}) \mp \left(i/\bar{k} \right) \mathbf{B}^{-1} \partial_z \phi(z,\underline{x}) \right],$$

$$\mathbf{B} = \left(K^2(z,\underline{x}) + \left(1/\bar{k}^2\right) \nabla_{\underline{x}}^2 \right)^{1/2}$$

Seismic Interferometry Modeling

While seismic interferometry uses the actual data, and is independent of an underlying velocity model, we will model the wave propagation process with the decoupled, one-way wave equations (lhs). Take appropriate one-way Green's functions for the seismic interferometry expression.

Seismic Interferometry

Goals

(1) Establish the connection between the signal processing CCF and PSF, and the experimental, one-sided TRM.

(2) Characterize the wave field components in the CCF and PSF (explicitly for a homogeneous medium).

(3) Understand the nature of the wave field components in terms of the TRI, governing, time-domain wave equation, the corresponding, one-sided TRM, and the non-TRI, one-way wave equation.

(4) Explicitly illustrate the connection between the PSF and the one-sided TRM for a homogeneous medium.

Connecting the CCF and TRM

$$TRM\left(t\right)=H\left(t\right)\int_0^{t^+} ds \int_{\mathbb{R}^2} d\underline{x}$$

$$\bullet G^-\left(t-s;z^Y,\underline{x}^Y;z,\underline{x}\right)G^+\left(T-s;z,\underline{x};z^Z,\underline{x}^Z\right)$$

Causal structure of Green's functions,

$$TRM\left(t\right)=\int_{-\infty}^{\infty} ds \int_{\mathbb{R}^2} d\underline{x}$$

$$\bullet G^-\left(t-s;z^Y,\underline{x}^Y;z,\underline{x}\right)G^+\left(T-s;z,\underline{x};z^Z,\underline{x}^Z\right)H\left(s\right)$$

Note that $TRM\left(t\right)=0, t<0$

Connecting the CCF and TRM

Fourier convolution theorem, causal structure of Green's function,

$$TRM(\omega) = \exp(i\omega T)\int_{\mathbb{R}^2} d\underline{x}\, G^{-}\left(\omega; z^{Y}, \underline{x}^{Y}; z, \underline{x}\right)$$

$$\bullet \int_0^T ds \exp(-i\omega s) G^{+}\left(s; z, \underline{x}; z^{Z}, \underline{x}^{Z}\right)$$

$$= \exp(i\omega T)\int_{\mathbb{R}^2} d\underline{x}\, G^{-}\left(\omega; z^{Y}, \underline{x}^{Y}; z, \underline{x}\right)$$

$$\bullet \int_{\mathbb{R}} d\omega' \frac{\sin\left[\left(\omega+\omega'\right)T\right]}{\pi\left(\omega+\omega'\right)} G^{+}\left(\omega'; z, \underline{x}; z^{Z}, \underline{x}^{Z}\right)$$

Connecting the CCF and TRM

Fourier transform back to time domain,

$$TRM(t) = \left(\frac{1}{2\pi}\right)\int_{\mathbb{R}} d\omega \exp\left[-i\omega(t-T)\right]\left\{\int_{\mathbb{R}^2} d\underline{x} G^{-}\left(\omega; z^Y, \underline{x}^Y; z, \underline{x}\right)\right.$$

$$\left. \bullet \int_{\mathbb{R}} d\omega' \frac{\sin\left[\left(\omega+\omega'\right)T\right]}{\pi\left(\omega+\omega'\right)} G^{+}\left(\omega'; z, \underline{x}; z^Z, \underline{x}^Z\right)\right\}$$

Take $\lim T \to \infty, t \to \infty, T - t \to \tau$, noting,

$$\frac{\sin\left[\left(\omega+\omega'\right)T\right]}{\pi\left(\omega+\omega'\right)} \xrightarrow[T\to\infty]{} \delta\left(\omega+\omega'\right) \qquad \text{(delta family)}$$

and

Connecting the CCF and TRM

$$G^{+}(-\omega)=\left[G^{+}(\omega)\right]^{*} \qquad \text{(symmetry relation)}$$

to obtain,

$$TRM(t)\xrightarrow[T\to\infty,t\to\infty,T-t\to\tau]{}\left(\frac{1}{2\pi}\right)\int_{\mathbb{R}}d\omega\exp(i\omega\tau)\left\{\int_{\mathbb{R}^2}d\underline{x}\right.$$

$$\left.\bullet G^{-}\left(\omega;z^{Y},\underline{x}^{Y};z,\underline{x}\right)\left[G^{+}\left(\omega;z,\underline{x};z^{Z},\underline{x}^{Z}\right)\right]^{*}\right\}$$

$$=CCF(-\tau)$$

Relationship holds for all choices of Green's function normalization

Green's Function Normalization

Relating the seismic 3-d VSP to the CCF in a form that connects directly to the one-sided TRM requires the application of s/r reciprocity. For a general, range- and laterally-dependent environment, the one-way Green's functions must satisfy the s/r reciprocity principle.

This can be accomplished with the vertical-acoustic-power-flux normalization. (The operator WKB amplitude.)

Since the calculations presented here are for range-independent environments, this is not an issue, since all normalized Green's functions satisfy the same one-way wave equation and the s/r reciprocity principle.

Calculation of Homogeneous Medium CCF

Specification of Green's functions (velocity = 1)

Want CCF to resemble a Green's function as closely as possible

$$G^{+}\left(\omega; z, \underline{x}; z^{Z}, \underline{x}^{Z}\right) = G\left(\omega; z, \underline{x}; z^{Z}, \underline{x}^{Z}\right) \quad \text{(point source GF)}$$

$$= \left(\frac{i\omega}{8\pi^{2}}\right) \int_{|\underline{p}| \leq 1} d\underline{p} \frac{\exp\left[i\omega\left(\underline{p} \bullet \left(\underline{x} - \underline{x}^{Z}\right) + \left(z - z^{Z}\right)\left(1 - |\underline{p}|^{2}\right)^{1/2}\right)\right]}{\left(1 - |\underline{p}|^{2}\right)^{1/2}}$$

$$+ \left(\frac{\omega}{8\pi^{2}}\right) \int_{|\underline{p}| \geq 1} d\underline{p} \frac{\exp\left[i\omega \underline{p} \bullet \left(\underline{x} - \underline{x}^{Z}\right)\right] \exp\left[-\omega\left(z - z^{Z}\right)\left(|\underline{p}|^{2} - 1\right)^{1/2}\right]}{\left(|\underline{p}|^{2} - 1\right)^{1/2}},$$

Calculation of Homogeneous Medium CCF

$$= G^{P}\left(\omega; z, \underline{x}; z^{Z}, \underline{x}^{Z}\right) + G^{NP}\left(\omega; z, \underline{x}; z^{Z}, \underline{x}^{Z}\right)$$

$$G^{-}\left(\omega; z^{Y}, \underline{x}^{Y}; z, \underline{x}\right) =$$

$$-\left(\frac{\omega}{2\pi}\right)^{2}\left\{\int_{|\underline{p}|\le 1} d\underline{p}\exp\left[i\omega\left(\underline{p}\bullet\left(\underline{x}^{Y}-\underline{x}\right)+\left(z-z^{Y}\right)\left(1-|\underline{p}|^{2}\right)^{1/2}\right)\right]\right.$$

$$\left.+\int_{|\underline{p}|\ge 1} d\underline{p}\exp\left[i\omega\,\underline{p}\bullet\left(\underline{x}^{Y}-\underline{x}\right)\right]\exp\left[-\omega\left(z-z^{Y}\right)\left(|\underline{p}|^{2}-1\right)^{1/2}\right]\right\}$$

(one-way propagator identically zero behind the mirror plane)

Substituting and evaluating results in,

$$CCF(\omega) = -\left[G^{P}\left(\omega; z^{Y}, \underline{x}^{Y}; z^{Z}, \underline{x}^{Z}\right)\right]^{*} - G^{NP}\left(\omega; \varsigma, \underline{x}^{Y}; 0, \underline{x}^{Z}\right),$$

Calculation of Homogeneous Medium CCF

$$= -G^{*}\left(\omega; z^{Y}, \underline{x}^{Y}; z^{Z}, \underline{x}^{Z}\right) + G^{NP}\left(\omega; z^{Y}, \underline{x}^{Y}; z^{Z}, \underline{x}^{Z}\right) - G^{NP}\left(\omega; \varsigma, \underline{x}^{Y}; 0, \underline{x}^{Z}\right),$$

$\varsigma = 2z - z^{Z} - z^{Y}$ ("total travel distance")

Same results if Green's functions chosen to be + and - one-way propagators with the vertical-acoustic-power-flux normalization

Fourier transforming to the time domain and evaluating the resulting integrals in standard fashion result in,

$$CCF(t) = -G\left(-t; z^{Y}, \underline{x}^{Y}; z^{Z}, \underline{x}^{Z}\right)$$

$$+ \frac{1}{(2\pi)^{2}\left(\left|\underline{\tilde{x}}\right|^{2} - t^{2}\right)_{+}^{1/2}} \left\{ \frac{\tilde{z}}{\left(\left|\underline{\tilde{x}}\right|^{2} - t^{2} + \tilde{z}^{2}\right)} - \frac{\varsigma}{\left(\left|\underline{\tilde{x}}\right|^{2} - t^{2} + \varsigma^{2}\right)} \right\},$$

Calculation of Homogeneous Medium CCF

$$CCF(t) = \left(\frac{-1}{4\pi\tilde{R}}\right)\delta\left(-t-\tilde{R}\right)$$

$$+\frac{1}{(2\pi)^2\left(\left|\underline{\tilde{x}}\right|^2-t^2\right)_+^{1/2}}\left\{\frac{\tilde{z}}{\left(\left|\underline{\tilde{x}}\right|^2-t^2+\tilde{z}^2\right)}-\frac{\varsigma}{\left(\left|\underline{\tilde{x}}\right|^2-t^2+\varsigma^2\right)}\right\},$$

$$\tilde{R}^2 = \tilde{z}^2 + \left|\underline{\tilde{x}}\right|^2, \tilde{z} = z^Y - z^Z, \underline{\tilde{x}} = \underline{x}^Y - \underline{x}^Z$$

$CCF(t)$ consists of:

(1) a cylindrical diffuse wave supported for $r = \left|\underline{\tilde{x}}\right| > \left|t\right|, t > 0$

(2) a spherical wave and the cylindrical diffuse wave for $t < 0$

When the receiver and source planes are the same, the CCF becomes the Point Spread Function (PSF)

Calculation of Homogeneous Medium CCF

$$\frac{\tilde{z}}{(2\pi)^2 \left(r^2 - t^2\right)_+^{1/2} \left(r^2 - t^2 + \tilde{z}^2\right)} \xrightarrow[\tilde{z}\to 0]{} \left(\frac{1}{4\pi}\right)\delta\left(r^2 - t^2\right) \quad \text{(delta family)}$$

then results in

$$PSF(t) = \left(\frac{1}{8\pi r}\right)\operatorname{sgn}(t)\,\delta\left(|t| - r\right) - \frac{\varsigma}{(2\pi)^2 \left(r^2 - t^2\right)_+^{1/2} \left(r^2 - t^2 + \varsigma^2\right)},$$

$$\varsigma = 2\left(z - z^Z\right)$$

Retaining only the propagating modes results in,

$$CCF^P(t) = \left(\frac{-1}{4\pi\tilde{R}}\right)\delta\left(-t - \tilde{R}\right) + \frac{\tilde{z}}{(2\pi)^2 \left(r^2 - t^2\right)_+^{1/2} \left(r^2 - t^2 + \tilde{z}^2\right)},$$

$$PSF^P(t) = \left(\frac{1}{8\pi r}\right)\operatorname{sgn}(t)\,\delta\left(|t| - r\right) = (1/2)\operatorname{sgn}(t)\,G\left(|t|\right)$$

Calculation of Homogeneous Medium TRM

Point source/receiver plane origin at (0, 0, 0)

Mirror plane origin at $(0, 0, x_3)$

$$TRM(t) = H(t)\int_0^{t^+} ds \int_{\mathbb{R}^2} d\underline{x}^m$$

$$\bullet G^-\left(t-s;0,\underline{x}^s;x_3,\underline{x}^m\right)G^+\left(T-s;x_3,\underline{x}^m;0,\underline{0}\right)$$

The function is symmetric about origin at (0, 0). Thus, take recording point at $(x_1^s, 0, 0)$, $x_1^s > 0$, and then take $x_1^s \rightarrow r$.

Fundamental properties

(1) $TRM(t) = 0, T < x_3$

(2) $TRM(t) = 0$, for sufficiently short times (hyperbolic)

Calculation of Homogeneous Medium TRM

(3) $TRM\left(t\right)$ is of finite extent in r on the source/receiver plane

(4) Despite 3-d nature of wave propagation, there will be weak (algebraic) singularities characteristic of 2-d case

Green's functions

$$G^{+}\left(T-s;x_3,\underline{x}^{m};0,\underline{0}\right)=\left(\frac{1}{4\pi L}\right)\delta\left(s-T+L\right),L^2=x_3^2+r^2$$

$$G^{-}\left(t-s;0,\underline{x}^{s};x_3,\underline{x}^{m}\right)=2\partial_{x_3}\left[\left(\frac{1}{4\pi \mathrm{L}}\right)\delta\left(t-s-\mathrm{L}\right)\right],$$

$$\mathrm{L}^2=x_3^2+r^2+\left(x_1^s\right)^2-2x_1^s r\cos\theta$$

transforming to polar coordinates in mirror plane

Calculation of Homogeneous Medium TRM

The derivative, ∂_{x_3}, is written in form,

$$2\partial_{x_3}\left[\left(\frac{1}{4\pi \mathrm{L}}\right)\delta\left(t-s-\mathrm{L}\right)\right]=\lim_{\xi_3\to x_3}\left\{2\partial_{\xi_3}\left[\left(\frac{1}{4\pi \mathrm{L}_\xi}\right)\delta\left(t-s-\mathrm{L}_\xi\right)\right]\right\},$$

$$\mathrm{L}_\xi^2=\xi_3^2+r^2+\left(x_1^s\right)^2-2x_1^s r\cos\theta$$

Substituting into the original expression results in,

$$TRM\left(t\right)=\left(\frac{1}{8\pi^2}\right)H\left(t\right)\lim_{\xi_3\to x_3}\left\{\partial_{\xi_3}\left[\int_0^{t^+}ds\int_0^{2\pi}d\theta\int_0^{\nu}drr\,\frac{\delta\left(t-s-\mathrm{L}_\xi\right)\delta\left(s-T+L\right)}{L\mathrm{L}_\xi}\right]\right\},$$

$$\nu=\left(T^2-x_3^2\right)^{1/2}$$

Evaluating the integrals in the above order results in,

Calculation of Homogeneous Medium TRM

$$TRM(t) = \left(\frac{1}{2\pi^2}\right) H(T-\tau) H(T-x_3) \lim_{\xi_3 \to x_3} \left\{ \partial_{\xi_3} \left[B(M) - B(0) \right] \right\},$$

$$B(M) = \mathrm{Re}\left\{ \alpha^{-1/2} \log\left[2\left(\alpha R(M)\right)^{1/2} + 2\alpha M + \beta \right] \right\},$$

$$R(M) = \alpha M^2 + \beta M + \gamma, M = \min\left(t, T - x_3\right), \tau = T - t,$$

and where α, β, γ are functions of r, τ, T, x_3, ξ_3

Evaluating the above expression results in,

$$TRM(t) = TRM(\tau, T) = \left(\frac{-x_3}{\pi^2}\right) H(T-\tau) H(T-x_3)$$

$$\bullet \left\{ \frac{\left(r^2 - \tau^2\right)T + 2\tau x_3^2}{\left(r^2 - \tau^2\right)_+^{1/2} \left(r^2 - \tau^2 + 4x_3^2\right) \left[\left(\tau^2 - r^2\right) \left(\left(r^2 - \tau^2\right) \left[r^2 - \left(2T - \tau\right)^2 \right] + 4r^2 x_3^2 \right) \right]_+^{1/2}} \right\}$$

Explicit Connection of PSF and TRM

For the same source and receiver plane,

$$TRM(t) = TRM(\tau, T) = \left(\frac{-x_3}{\pi^2}\right) H(T-\tau) H(T-x_3)$$

$$\bullet \left\{ \frac{\left(r^2-\tau^2\right)T + 2\tau x_3^2}{\left(r^2-\tau^2\right)_+^{1/2}\left(r^2-\tau^2+4x_3^2\right)\left[\left(\tau^2-r^2\right)\left(\left(r^2-\tau^2\right)\left[r^2-\left(2T-\tau\right)^2\right]+4r^2x_3^2\right)\right]_+^{1/2}} \right\}$$

In the $\lim T \to \infty$, there are two terms; the first,

$$\frac{\left(-x_3/\pi^2\right)\left(r^2-\tau^2\right)T}{\left(r^2-\tau^2\right)_+^{1/2}\left(r^2-\tau^2+4x_3^2\right)\left[\left(\tau^2-r^2\right)\left(\left(r^2-\tau^2\right)\left[r^2-\left(2T-\tau\right)^2\right]+4r^2x_3^2\right)\right]_+^{1/2}}$$

and the second,

Explicit Connection of PSF and TRM

$$\frac{\left(-2\tau x_3^3/\pi^2\right)}{\left(r^2-\tau^2\right)_+^{1/2}\left(r^2-\tau^2+4x_3^2\right)\left[\left(\tau^2-r^2\right)\left(\left(r^2-\tau^2\right)\left[r^2-\left(2T-\tau\right)^2\right]+4r^2x_3^2\right)\right]_+^{1/2}}$$

In the infinite T limit, the first term goes to,

$$Term1 \xrightarrow[T\to\infty]{} \frac{-\varsigma}{\left(2\pi\right)^2\left(r^2-\tau^2\right)_+^{1/2}\left(r^2-\tau^2+\varsigma^2\right)}$$

where $\varsigma = 2x_3$ is the appropriate travel distance

Retaining the leading order term and noting that,

$$f_\Delta\left(x\right)=\frac{\Delta}{\pi x\left(x-\Delta^2\right)_+^{1/2}} \quad , \qquad \Delta \to 0 \quad ,$$

is a delta family, establishes that the second term goes to,

Explicit Connection of PSF and TRM

$$Term2 \xrightarrow[T\to\infty]{} \left(\frac{-\tau}{4\pi r}\right)\delta\left(r^2-\tau^2\right)$$

$$= \left(\frac{-1}{8\pi r}\right)\operatorname{sgn}(\tau)\,\delta\left(|\tau|-r\right)$$

Combining the two results then establishes that,

$$TRM(t) = TRM(\tau, T) \xrightarrow[T\to\infty]{} \left(\frac{-1}{8\pi r}\right)\operatorname{sgn}(\tau)\,\delta\left(|\tau|-r\right)$$

$$-\frac{\varsigma}{(2\pi)^2\left(r^2-\tau^2\right)_+^{1/2}\left(r^2-\tau^2+\varsigma^2\right)}$$

$$= PSF(-\tau)$$

Observations

(1) In addition to the expected spherical wave Green's function components, the CCF and PSF contain cylindrical diffuse wave components.

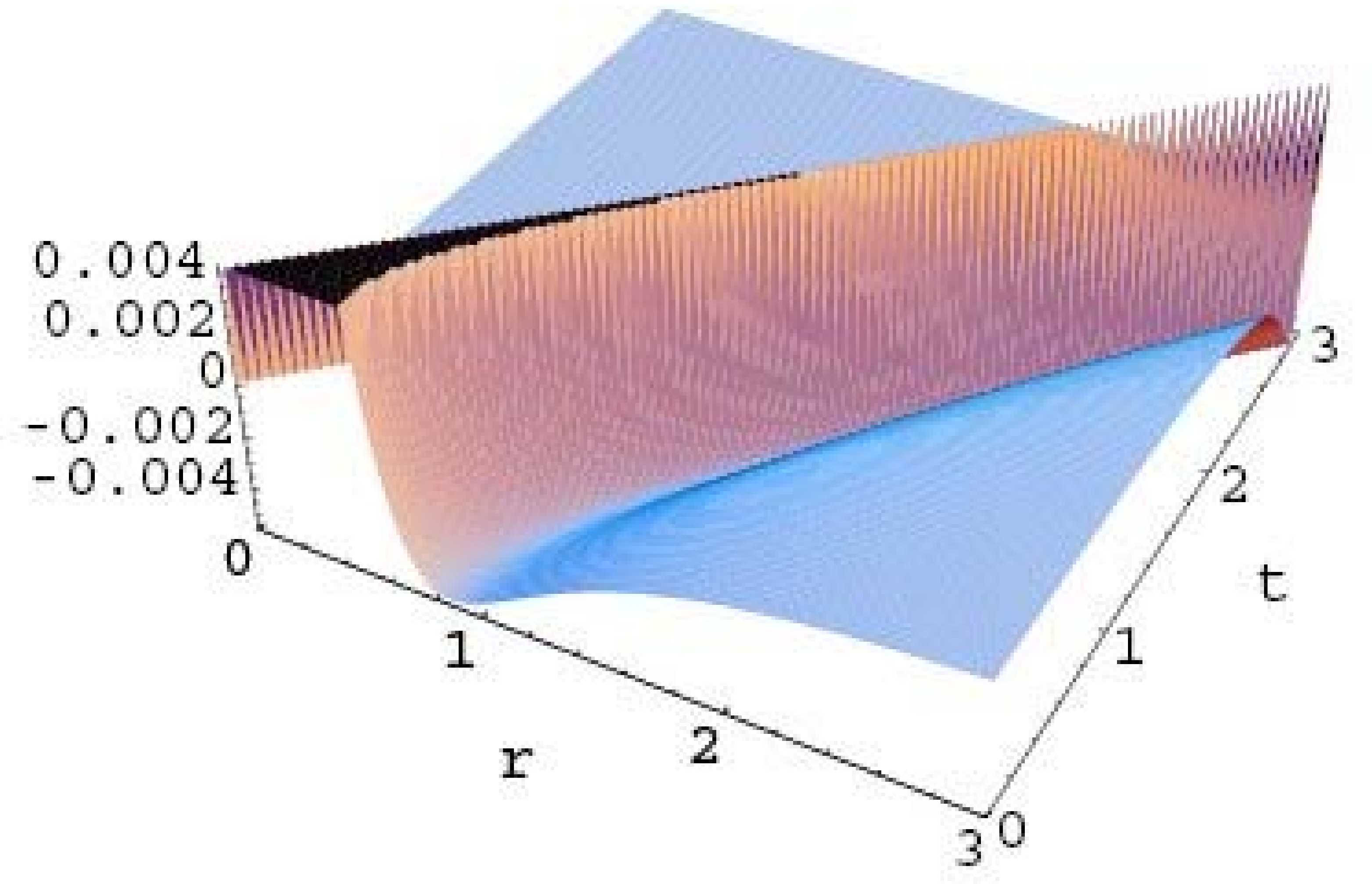
(2) In the CCF, one of the cylindrical diffuse wave components originates with the propagating modes.

(3) Even restricting the calculations to the propagating modes, and restoring TRI at the level of the one-way wave equation, the PSF displays an "extraneous" Green's function component.

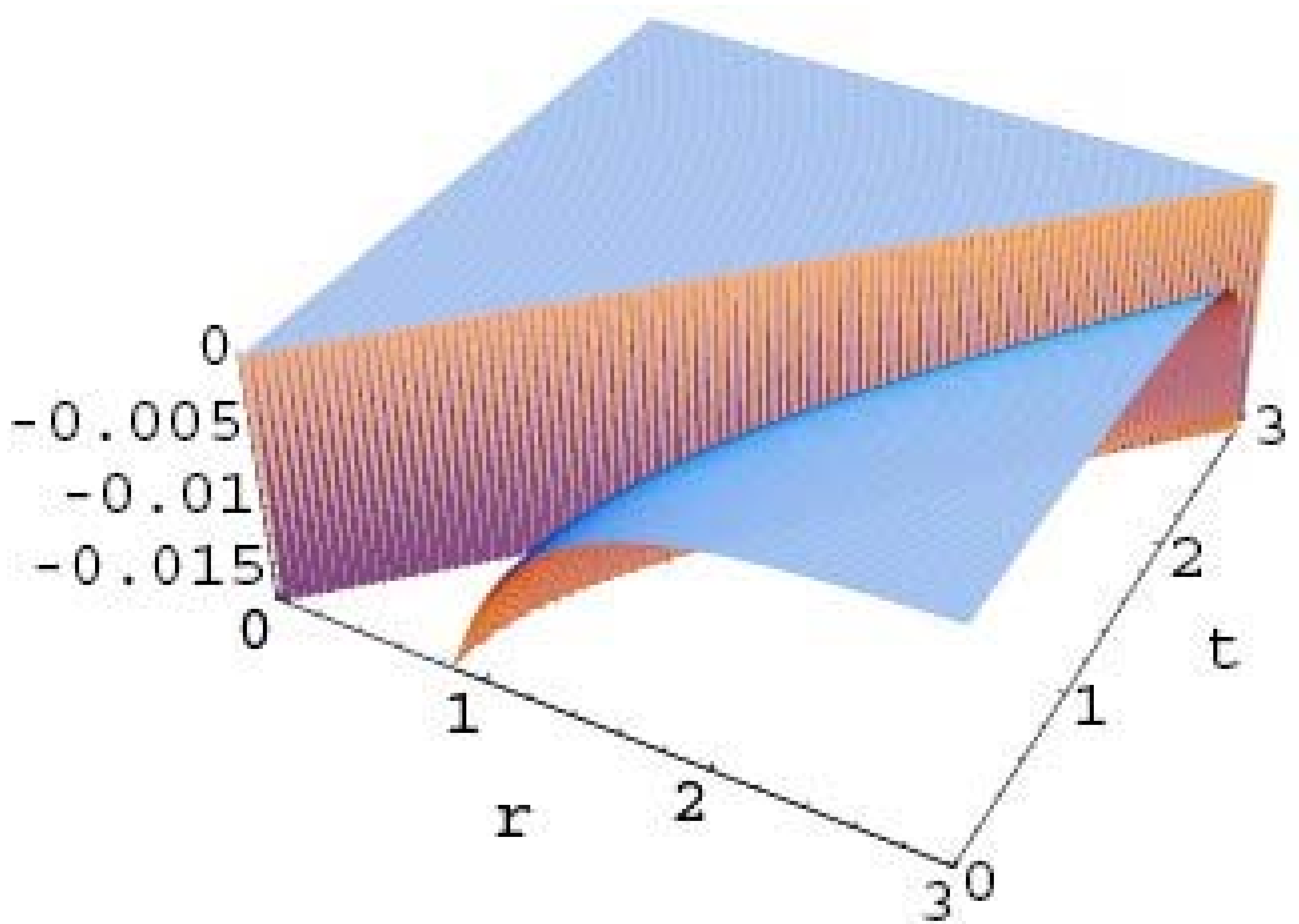
(4) This is consistent with the TRI, governing, time-domain wave equation, the corresponding, one-sided TRM, and the non-TRI one-way wave equation.

(5) The one-sided TRM is consistent with a finite propagation speed; the CCF and PSF do not correspond to finite propagation speed, one-sided TRMs.

Homogeneous Medium CCF
Diffuse Wave $\left(z^{Y}-z^{Z}=1/3, \varsigma=1\right)$

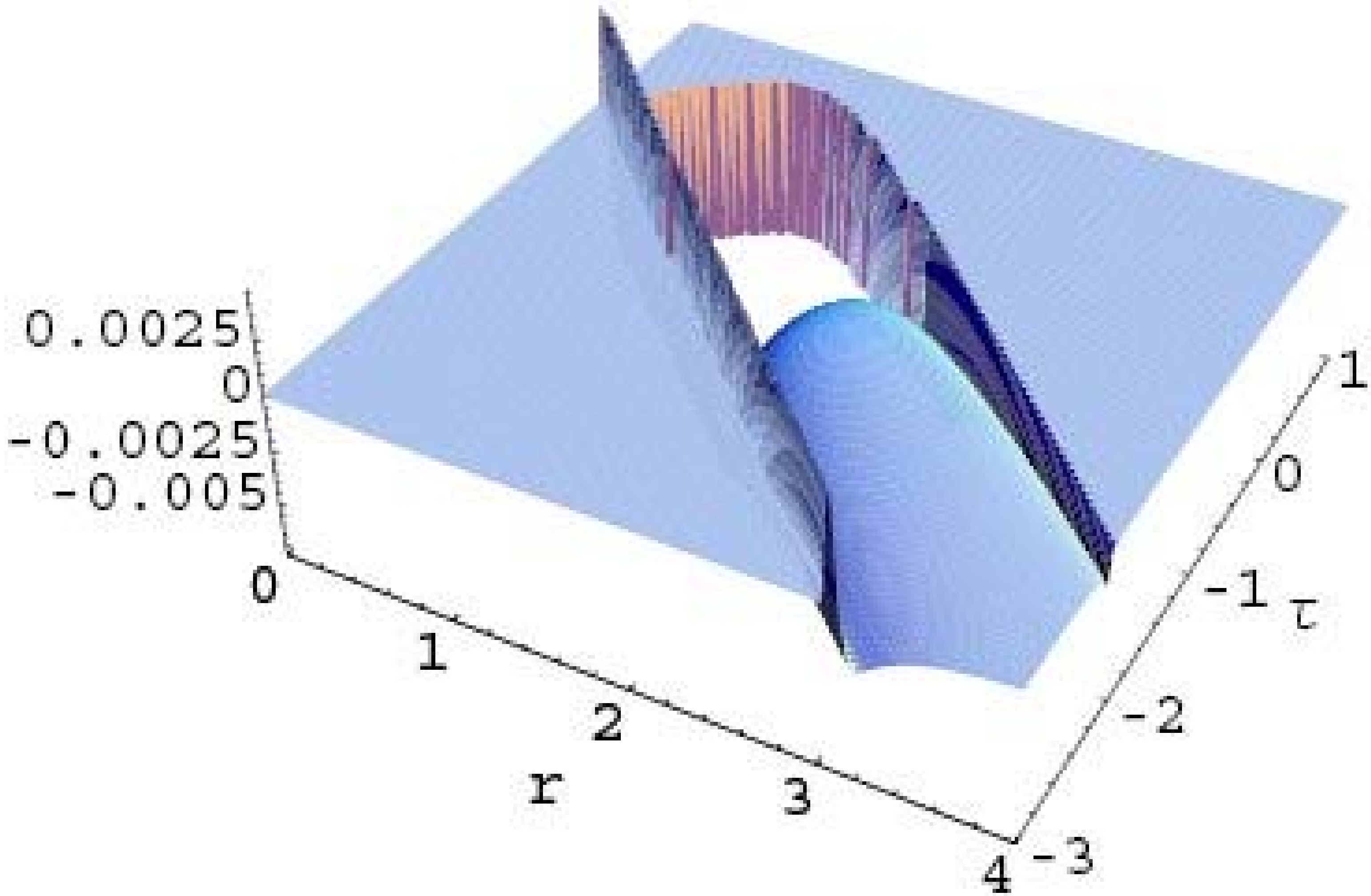


Homogeneous Medium PSF
Diffuse Wave $(\varsigma = 2/3)$

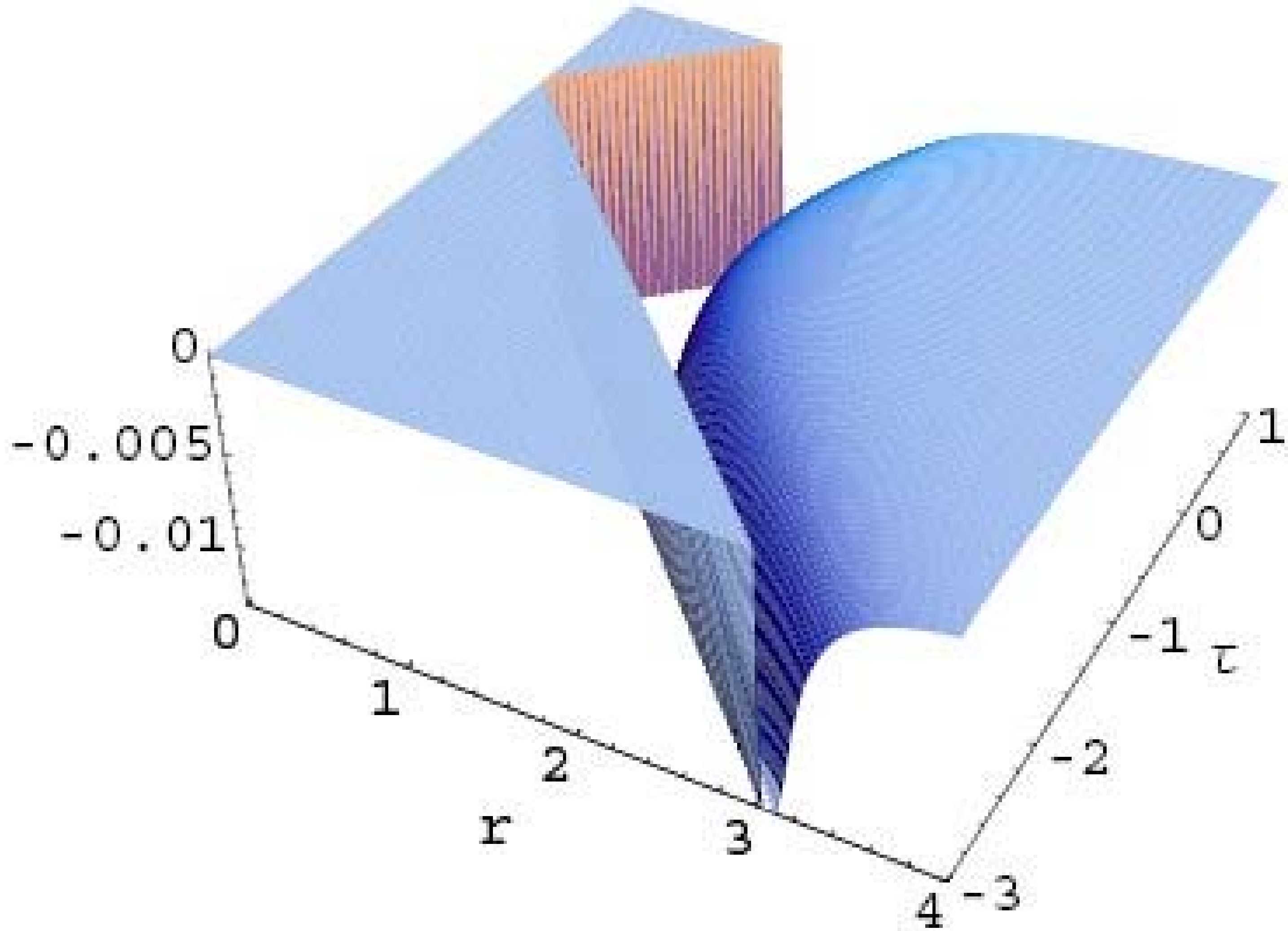


Homogeneous Medium TRM

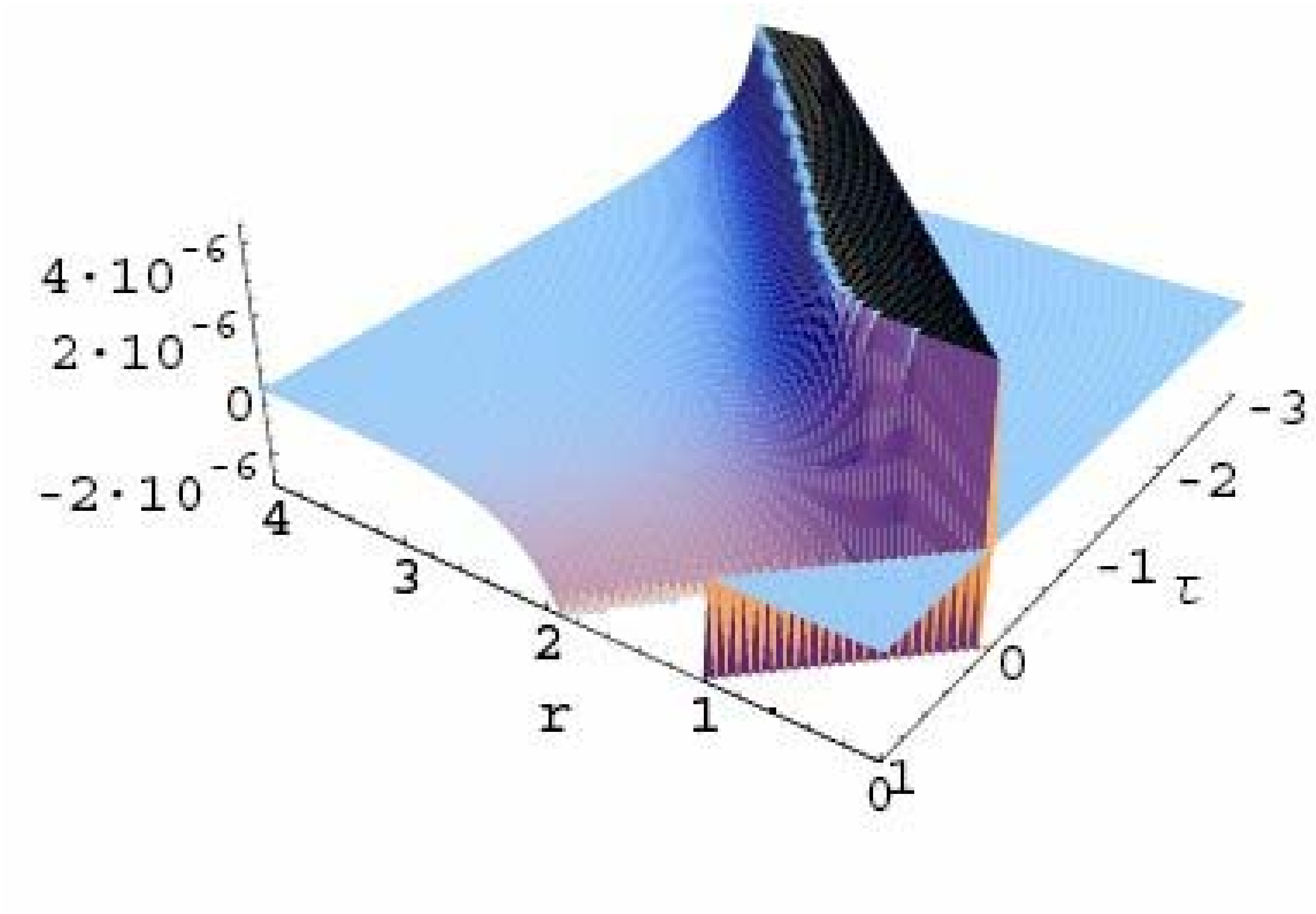
$$\left(T = 1, x_3 = 1/3\right)$$



Homogeneous Medium TRM
First Term $\left(T = 100, x_3 = 1/3\right)$



Homogeneous Medium TRM Second Term $\left(T = 100, x_3 = 1/3\right)$



Interferometry with a Reflection: Interface Profile

2-d, discontinuity (two-layer) profile,

$$K^2(x) = K_1^2 + \left(K_2^2 - K_1^2\right) H(x), K_1, K_2 > 0$$

Point Spread Function (propagating modes only)

$$PSF^P(t) = (1/2)\,\mathrm{sgn}(t)\, G\left(|t|; 0, x^r; 0, x^s\right)$$

Defining

$$x_> = \max\left(x^r, x^s\right), x_< = \min\left(x^r, x^s\right),$$

there are three distinct cases:

(1) $x_> \geq 0 \geq x_<$ (transmission problem)

(2) $x_> \geq x_< \geq 0$ (reflection problem)

(3) $x_< \leq x_> \leq 0$ (reflection problem)

Interferometry with a Reflection: Interface Profile

Case (1): $x_> \geq 0 \geq x_<$

$$PSF^P(t) = (1/2)\,\mathrm{sgn}(t)\,H\left[c_0|t| - \left(K_2 x_> + K_1|x_<|\right)\right]$$

$$\bullet \left(\frac{c_0}{\pi\left(K_1^2 - K_2^2\right)}\right)\left(\frac{\gamma_1\gamma_2\left(\gamma_1 - \gamma_2\right)}{\zeta\left(\gamma_1 x_> + \gamma_2|x_<|\right)}\right)\Bigg|_{\zeta=\zeta_0}, \qquad \gamma_{1,2} = \left(K_{1,2}^2 + \zeta^2\right)^{1/2},$$

$$\zeta_0 = \left[\frac{1}{\left(x_>^2 - x_<^2\right)^2}\left(\frac{x_>^2 + x_<^2}{2x_>^2}\right)\mu - \left(\frac{K_2^2 x_>^2 - K_1^2 x_<^2}{x_>^2 - x_<^2}\right)\right.$$

$$\left. - \frac{1}{\left(x_>^2 - x_<^2\right)^2}\left[\left(\frac{x_<^2}{x_>^2}\right)\mu^2 + 2x_<^2\left(x_<^2 - x_>^2\right)\left(K_2^2 - K_1^2\right)\mu\right]^{1/2}\right]^{1/2},$$

$$\mu = 2\left(c_0 t x_>\right)^2$$

Interferometry with a Reflection: Interface Profile

Case (2): $x_> \ge x_< \ge 0$

$$PSF^P(t) = (1/2)\,\mathrm{sgn}(t)\left\{\left(\frac{c_0}{2\pi\left[(c_0 t)^2 - (K_2 R)^2\right]^{1/2}}\right) H\left[c_0|t| - K_2 R\right]\right.$$

$$\left. + H\left[c_0|t| - K_2\rho\right]\left(\frac{c_0}{2\pi\left(K_2^2 - K_1^2\right)\left[(c_0 t)^2 - (K_2\rho)^2\right]^{1/2}}\right)\left[\frac{c_0|t|}{\rho} - \left(\left(\frac{c_0 t}{\rho}\right)^2 + \left(K_1^2 - K_2^2\right)\right)^{1/2}\right]^2\right\},$$

$$R = x_> - x_< = \left|x^r - x^s\right|, \rho = x_> + x_< = x^r + x^s$$

Case (3): $x_< \le x_> \le 0$

The same result as case (2) upon interchanging K_1, K_2

and noting that $\rho = \left|x_>\right| + \left|x_<\right| = \left|x^r\right| + \left|x^s\right|$

Summary

(1) The CCF and PSF are signal processing constructions, and the one-sided TRM is an experiment; they are related by,

$$TRM\left(t\right)\xrightarrow[T\to\infty,t\to\infty,T-t\to\tau]{}CCF\left(-\tau\right)$$

(2) The CCF and PSF contain components other than the expected Green's function components. (From a naive perspective, these are viewed as "artifacts.")

(3) This is consistent with the TRI, governing, time-domain wave equation, the corresponding, one-sided TRM, and the non-TRI one-way wave equation.

(4) Even restricting the calculations to the propagating modes, and restoring TRI at the level of the one-way wave equation, there are "artifacts" as viewed from the naive perspective.