A Hitchhiker's Guide to the Seismic Phase Space and Path Integral Universe

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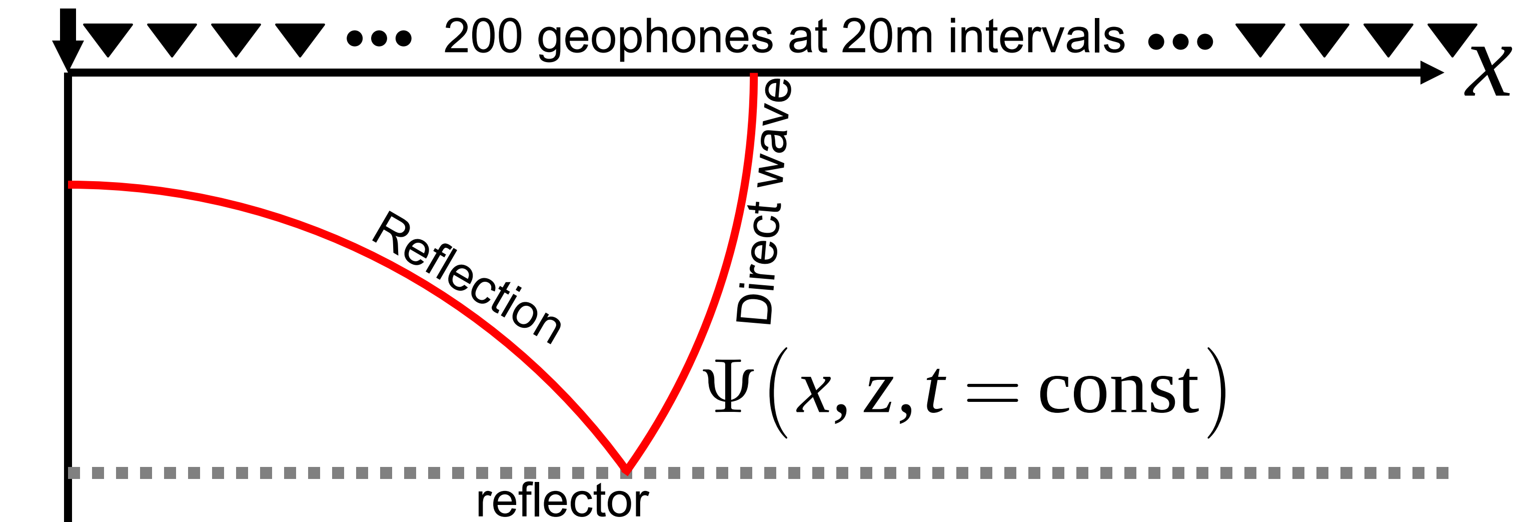
POTSI
PSEUDODIFFERENTIAL OPERATOR THEORY & SEISMIC IMAGING

Pacific Institute for the Mathematical Sciences

Seismic Shot Record



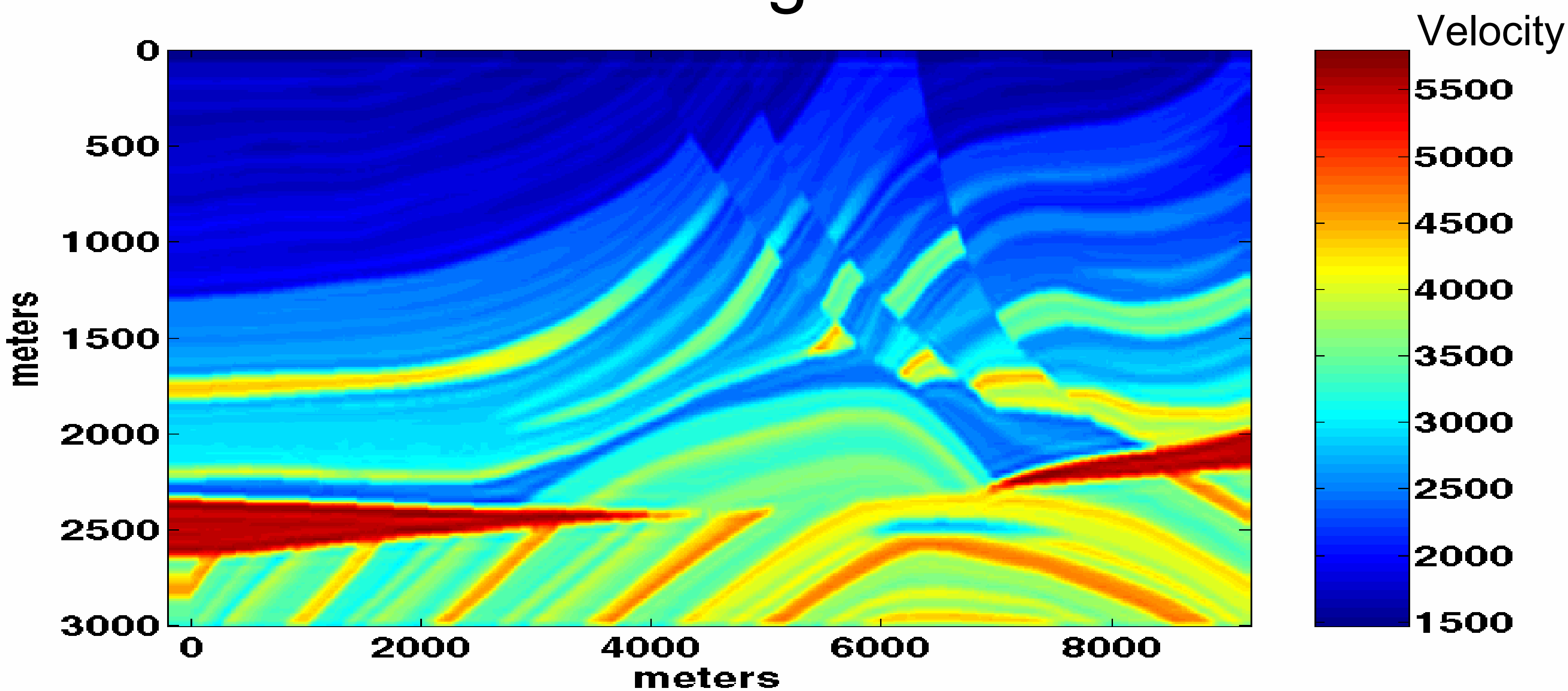
Geophones record $\Psi\left(x, z = 0, t\right) \in \mathbb{R} \times \mathbb{R}$

Want to know:

- Positions of reflectors
- Numerical estimates of earth properties

Classical Wave Theory

Environmental Modeling: Marmousi



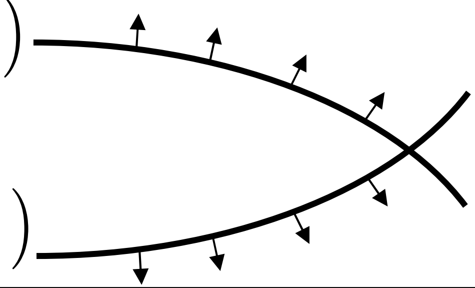
Environmental Difficulties

1) Complex, layered environments
2) Multidimensional environments
3) Inhomogeneous background
4) Large scale (many wavelengths)
5) Strongly inhomogeneous environments
6) Focusing and defocusing regimes

Seismic Imaging Paradigm

A common seismic imaging methodology is derivable from first-order inverse Born scattering

$$\Psi_{refl}\left(\vec{x},t_{inc}\right)=r\left(\vec{x}\right)\Psi_{inc}\left(\vec{x},t_{inc}\right)$$



$$\Psi_{\mathrm{inc}}\left(\vec{x},t_{inc}\right)$$

reflector

$$\frac{\Psi_{refl}\left(\vec{x},t_{inc}\right)}{\Psi_{inc}\left(\vec{x},t_{inc}\right)}=r\left(\vec{x}\right)$$ A reflectivity estimate.

Seismic Imaging Paradigm

Seismic imaging typically is done in the frequency domain and uses depth steps not time steps, so a more common imaging condition is:

$$r(x, y, \Delta z) = \sum_k \frac{\psi_{refl}(x, y, z = \Delta z, k\Delta\omega)}{\psi_{inc}(x, y, z = \Delta z, k\Delta\omega)}$$

Locally Homogeneous Medium Wavefield Extrapolation
(the GPSPI method)

$$\psi^+(x, z+\Delta z, \omega) \approx \int_{\mathbb{R}} dk_x e^{i\Delta z\sqrt{\frac{\omega^2}{v(x)^2} - k_x^2}} e^{ik_x x}\,\hat{\psi}^+(k_x, z, \omega)$$

In the limit of an infinitesimal step, the corresponding one-way wave equation is

$$i\partial_z\psi^+(x,z) + \frac{1}{2\pi}\int_{\mathbb{R}^2} dk_x dx' \left(\frac{\omega^2}{v(x)^2} - k_x^2\right)^{1/2} \mathrm{e}^{ik_x(x-x')}\,\psi^+(x',z) = 0$$

Locally Homogeneous Medium Wavefield Extrapolation (GPSPI)
(physics formulation)

$$\phi^{+}\left(x+\Delta x, z\right) \approx \int_{\mathbb{R}} dp \exp\left(i\bar{k}pz\right)\left[\exp\left(i\bar{k}\Delta x\left(K^{2}\left(z\right)-p^{2}\right)^{1/2}\right)\hat{\phi}^{+}\left(x, p\right)\right]$$

$$\left(i/\bar{k}\right)\partial_{x}\phi^{+}\left(x,z\right)+\frac{\bar{k}}{2\pi}\int_{\mathbb{R}^{2}} dpdz'\left(K^{2}\left(z\right)-p^{2}\right)^{1/2} \bullet \exp\left(i\bar{k}p\left(z-z'\right)\right)\phi^{+}\left(x,z'\right)=0$$

$$K\left(z\right)=\frac{c_{0}}{c\left(z\right)} \qquad \bar{k}=\frac{\omega}{c_{0}} \qquad c\left(z\right)=v\left(z\right)$$

Three Common Misconceptions About GPSPI

1) The one-way wave equation corresponding to the limiting form of the GPSPI algorithm

$$\left(i/\bar{k}\right)\partial_x\phi^+(x,z)+\frac{\bar{k}}{2\pi}\int_{\mathbb{R}^2}dp\,dz'\overbrace{\left(K^2(z)-p^2\right)^{1/2}}^{\text{symbol}}$$

$$\bullet\exp\left(i\bar{k}p\left(z-z'\right)\right)\phi^+\left(x,z'\right)=0$$

is believed exact for a range-independent medium. The square-root function is believed to be the correct function (symbol) for infinitesimal wavefield extrapolation.

Three Common Misconceptions About GPSPI

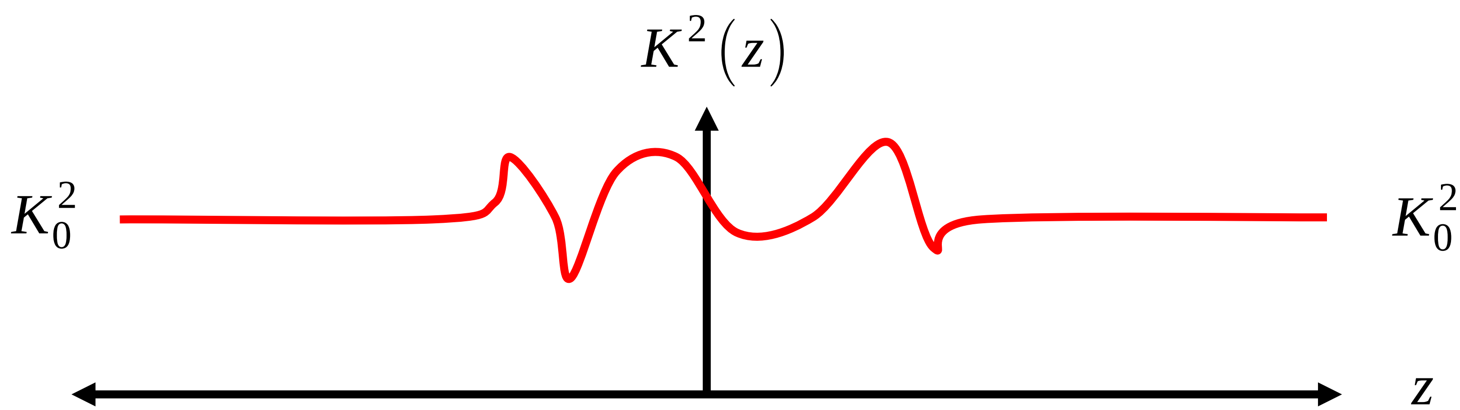
2) The wavefield growth problems, which can develop for finite, range step-size, are believed to vanish in the limit of zero, range step-size, since, in that limit, the theory is believed to be exact. The amplitude problems are assumed a result of the numerical discretization, and are not viewed as fundamental in nature.

3) Since, in the typical derivation of GPSPI, the up- and down-going wavefields are assumed to be independent, it is thought to be impossible to extrapolate a full, two-way wavefield by well-posed, one-way marching methods.

Misconception 1

Physical arguments (and rigorous mathematical arguments) establish that the symbol must be frequency dependent.

Consider the following environment:



in the (1) high- and (2) low-frequency limits.

Misconception 1

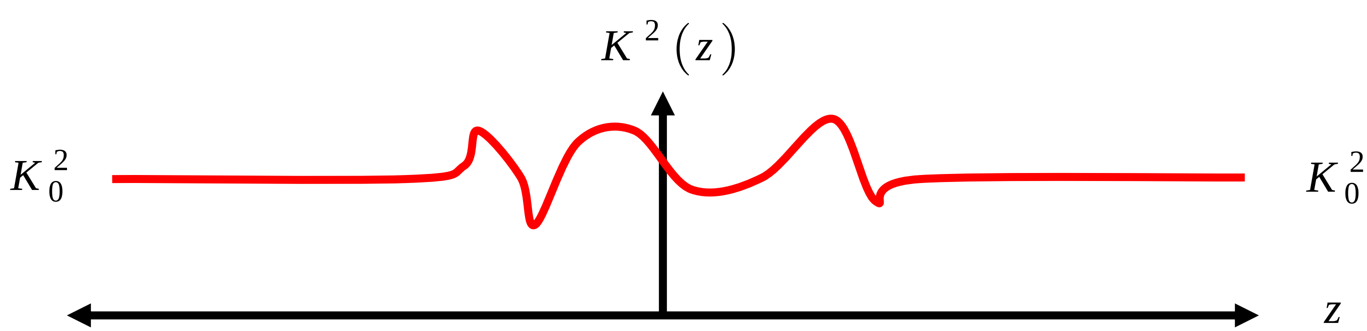
(1) In the high-frequency limit, the environment appears locally homogeneous, and the exact symbol approaches

$$\left(K^2(z) - p^2\right)^{1/2}$$

(2) In the low-frequency limit, the environment appears globally homogeneous, and the exact symbol approaches

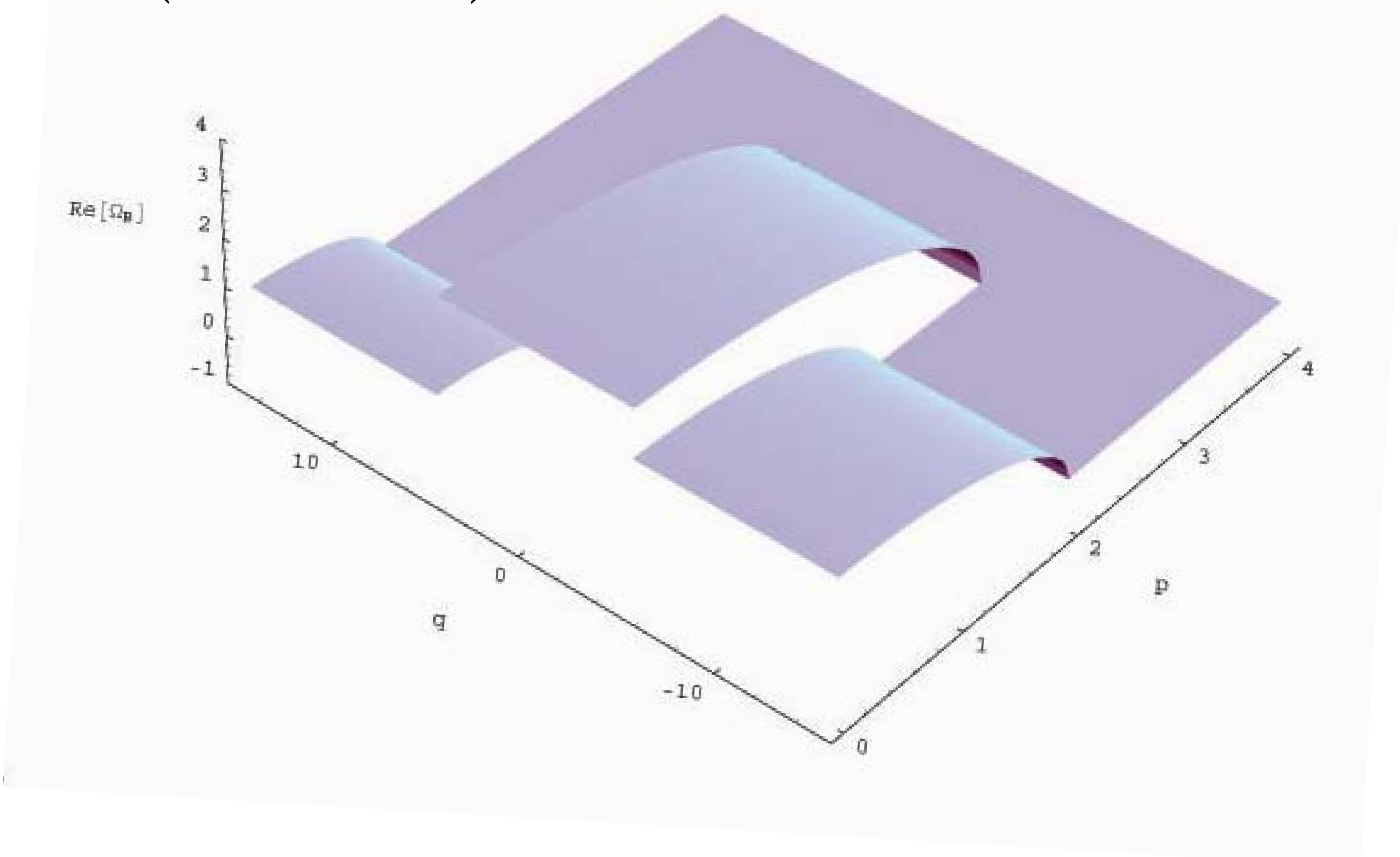
$$\left(K_0^2 - p^2\right)^{1/2}$$

Thus, the exact symbol must be frequency dependent, and cannot be the locally homogeneous symbol for all frequencies.



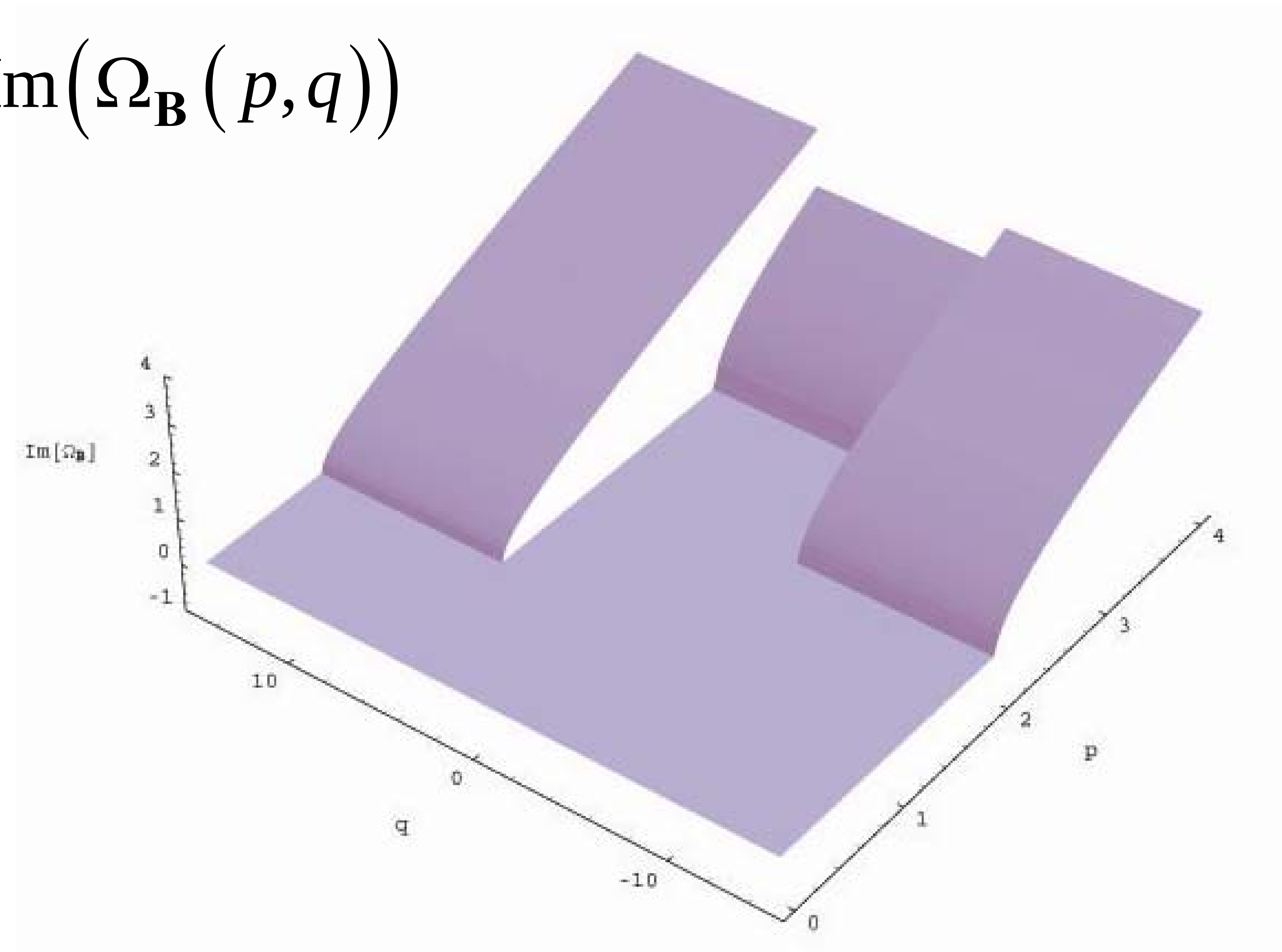
Locally Homogeneous Approximation
3-Layer Profile

$$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



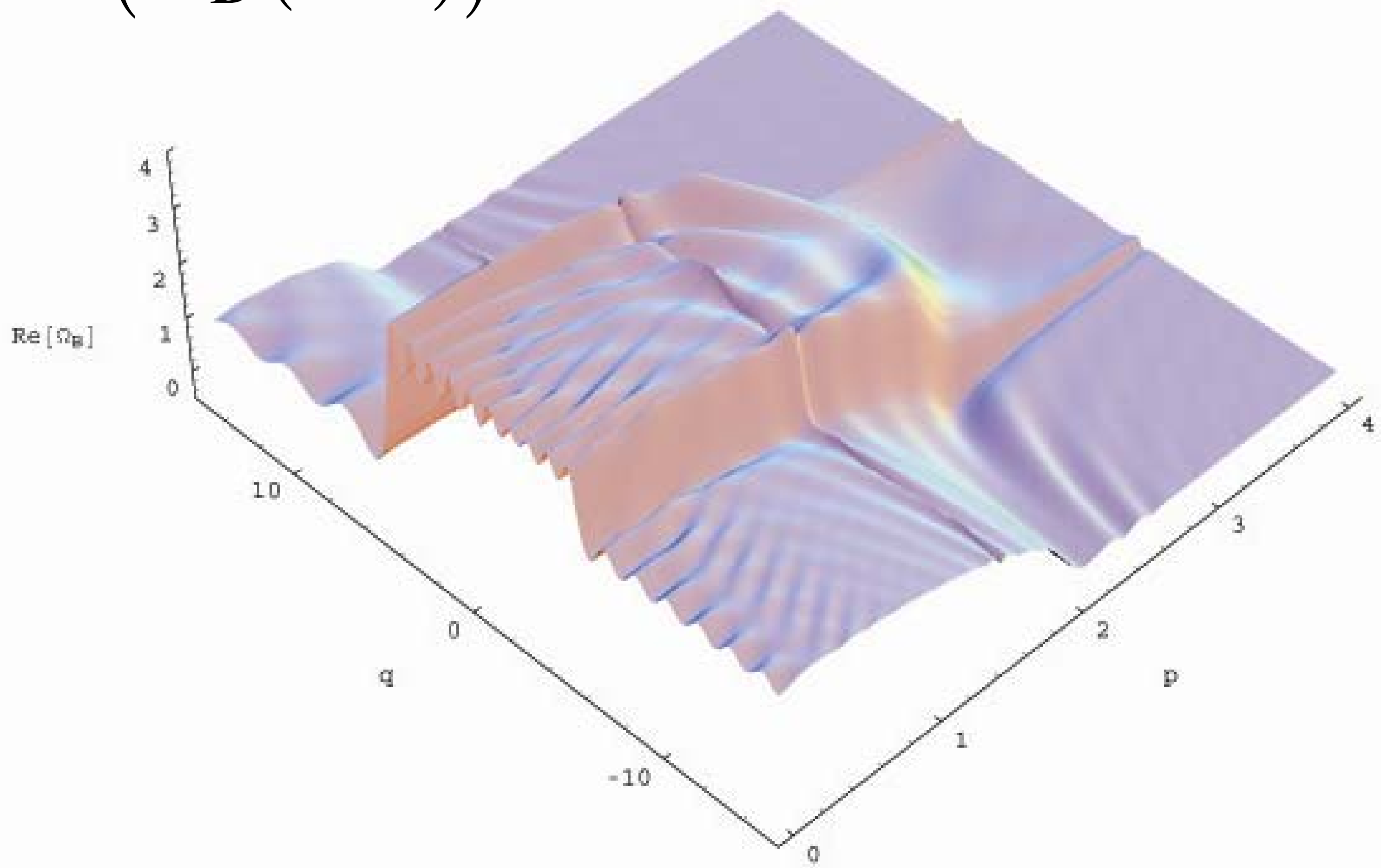
Locally Homogeneous Approximation
3-Layer Profile

$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



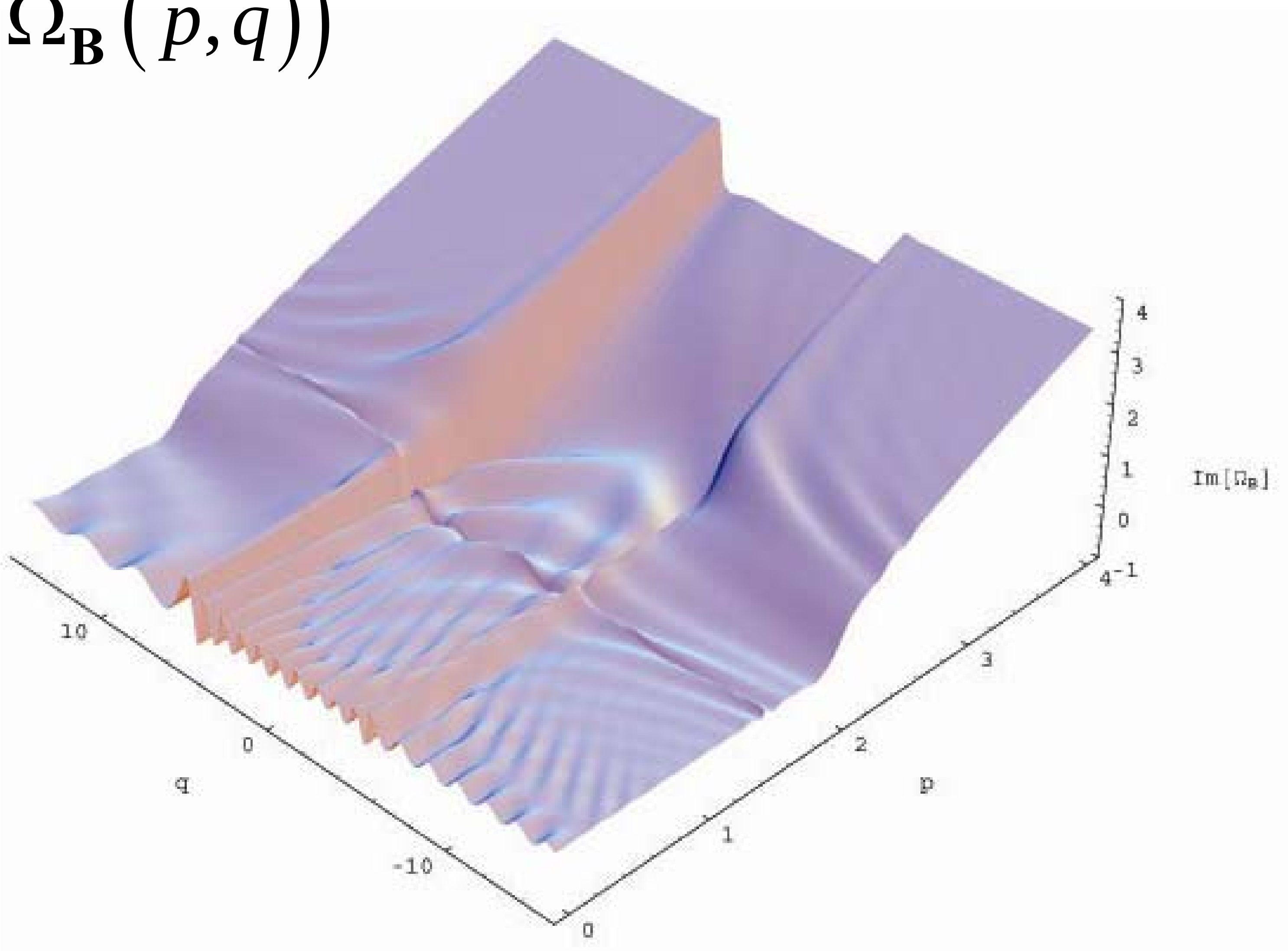
Exact Operator Symbol
3-Layer Profile

$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



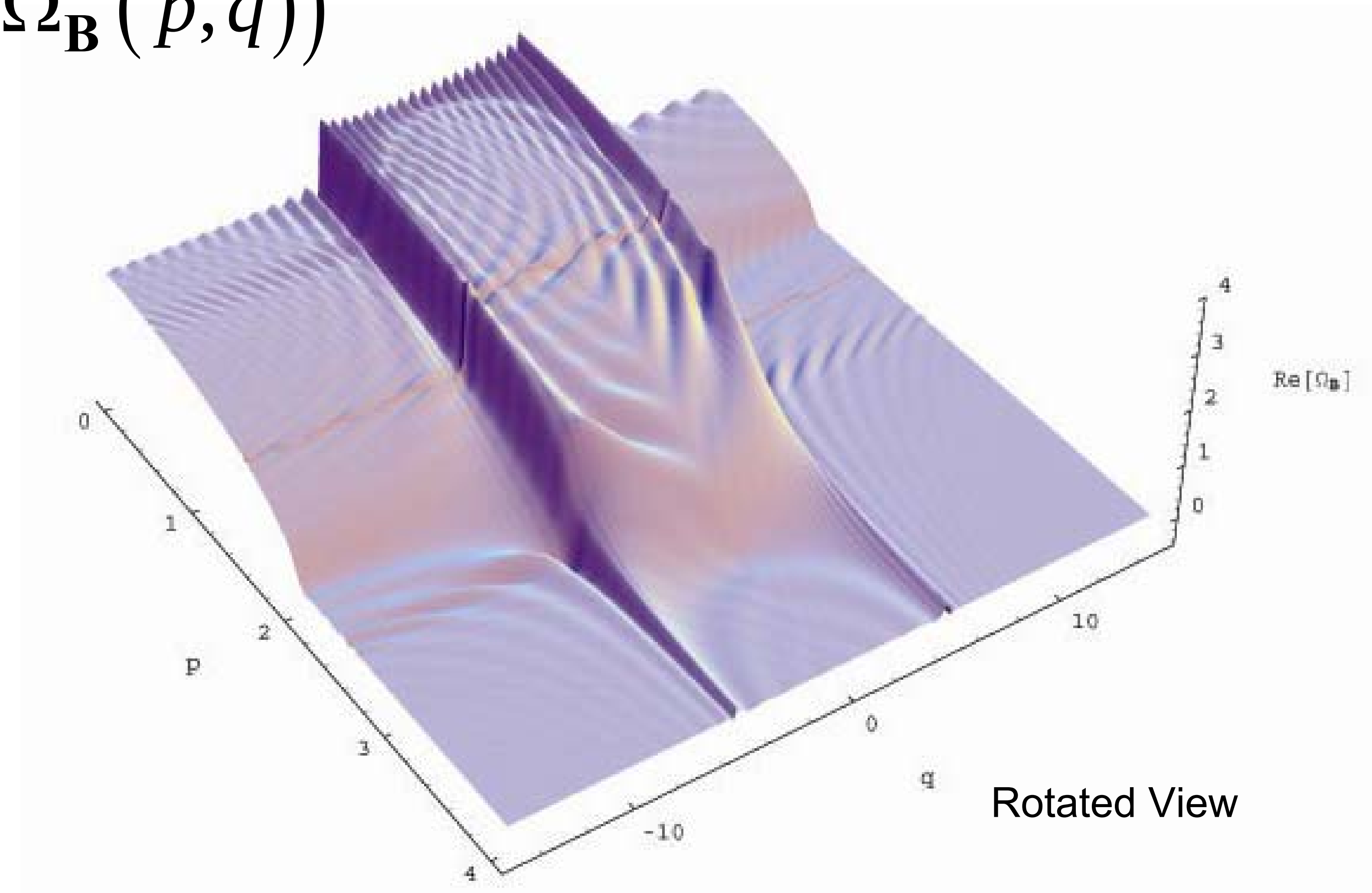
Exact Operator Symbol
3-Layer Profile

$\text{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Exact Operator Symbol
3-Layer Profile

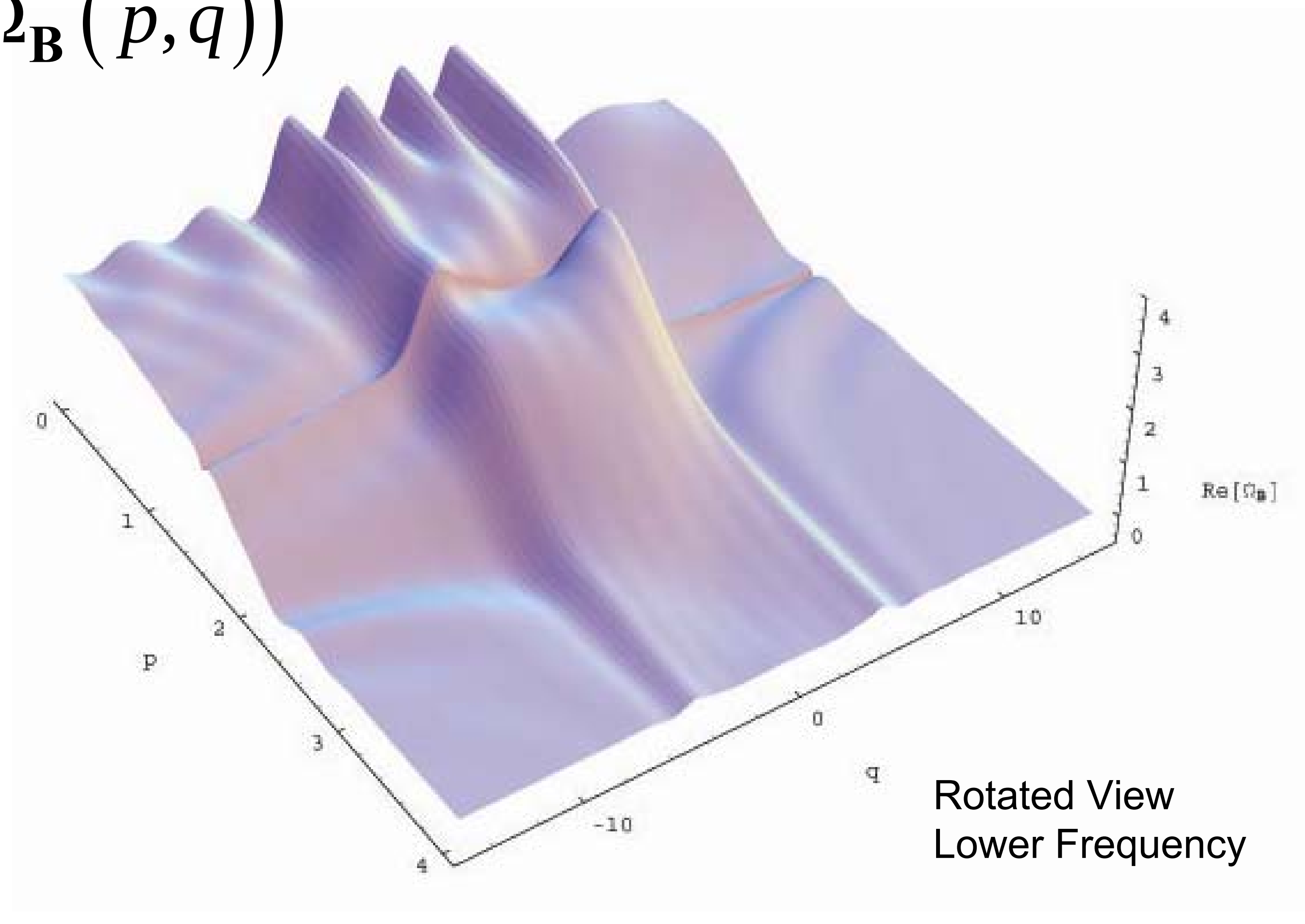
$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Rotated View

Exact Operator Symbol
3-Layer Profile

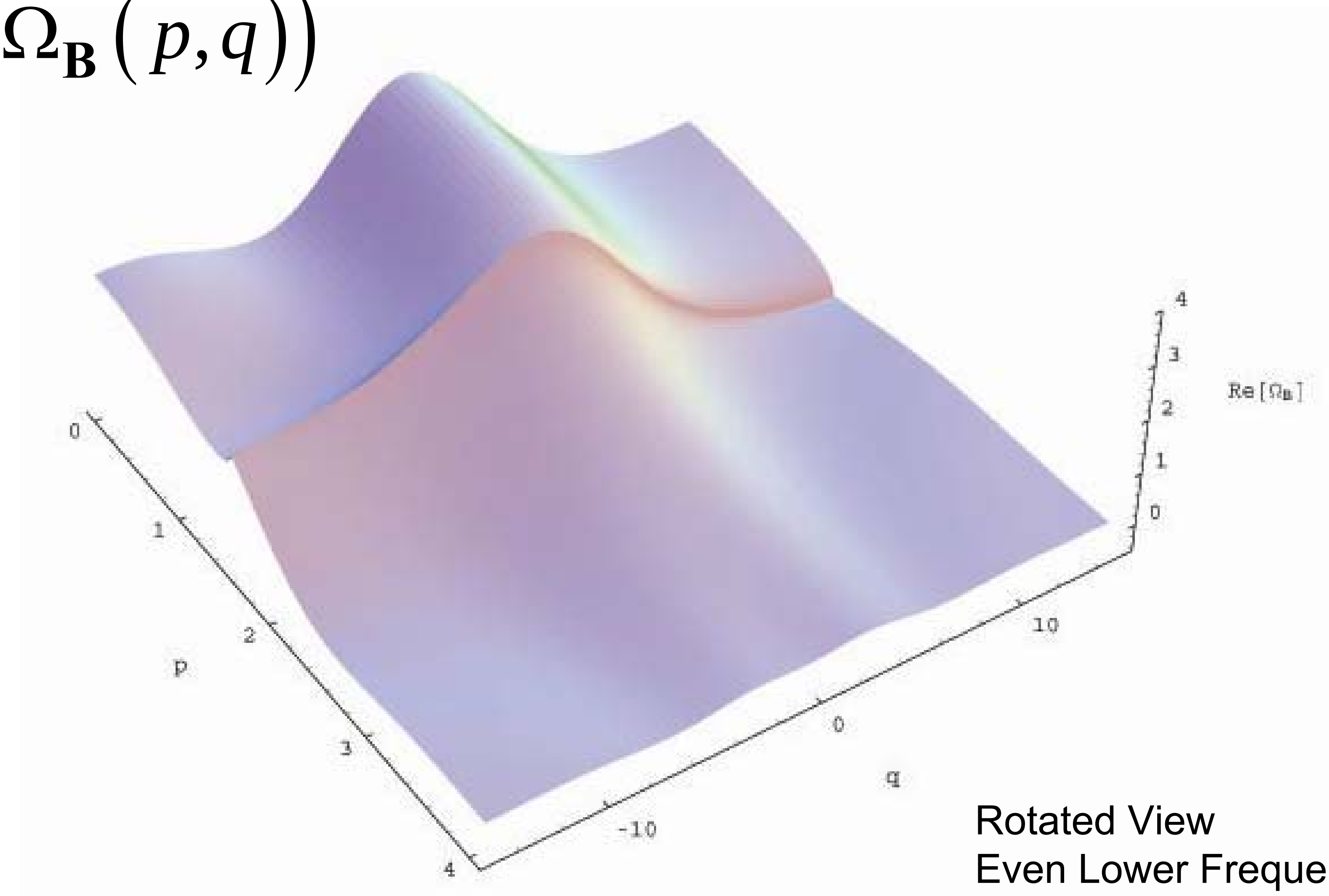
$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Rotated View
Lower Frequency

Exact Operator Symbol
3-Layer Profile

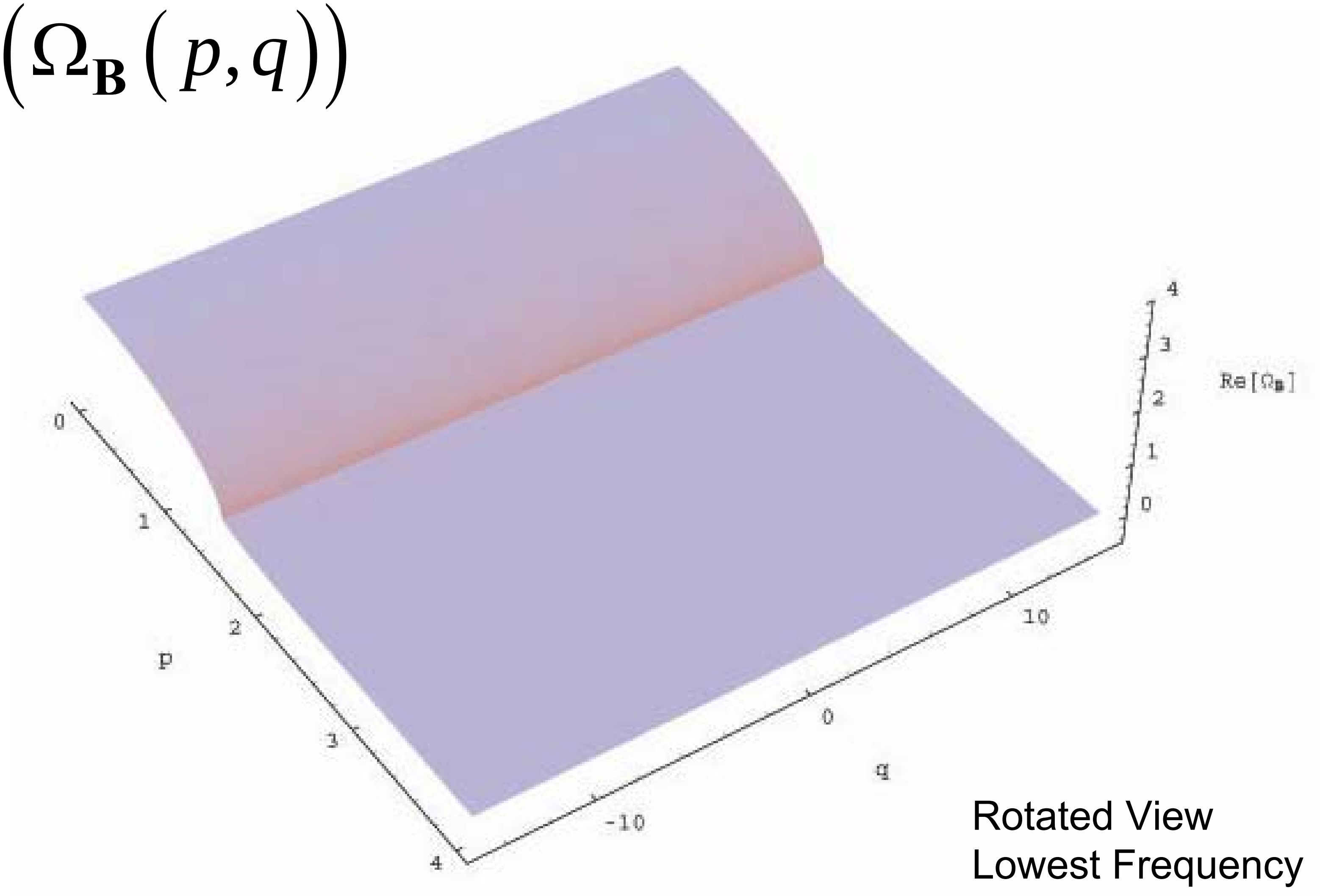
$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Rotated View
Even Lower Frequency

Exact Operator Symbol
3-Layer Profile

$\text{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Rotated View
Lowest Frequency

Misconceptions 2 and 3

2) The locally homogeneous square-root function is just an approximation to the exact symbol. It does not inherently conserve the integrated energy flux. The wavefield growth problems experienced by GPSPI are fundamental, and not solely a discretization artifact. This is a reflection of the true nature of this APPROXIMATION.

3) We will, indeed, construct an exact, well-posed, one-way reformulation of the two-way Helmholtz equation.

Classical Wave Theory

Modern Approaches – Principal Themes

1) Incorporation of well-posed, one-way methods into inherently two-way, global formulations

2) Exploitation of correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics

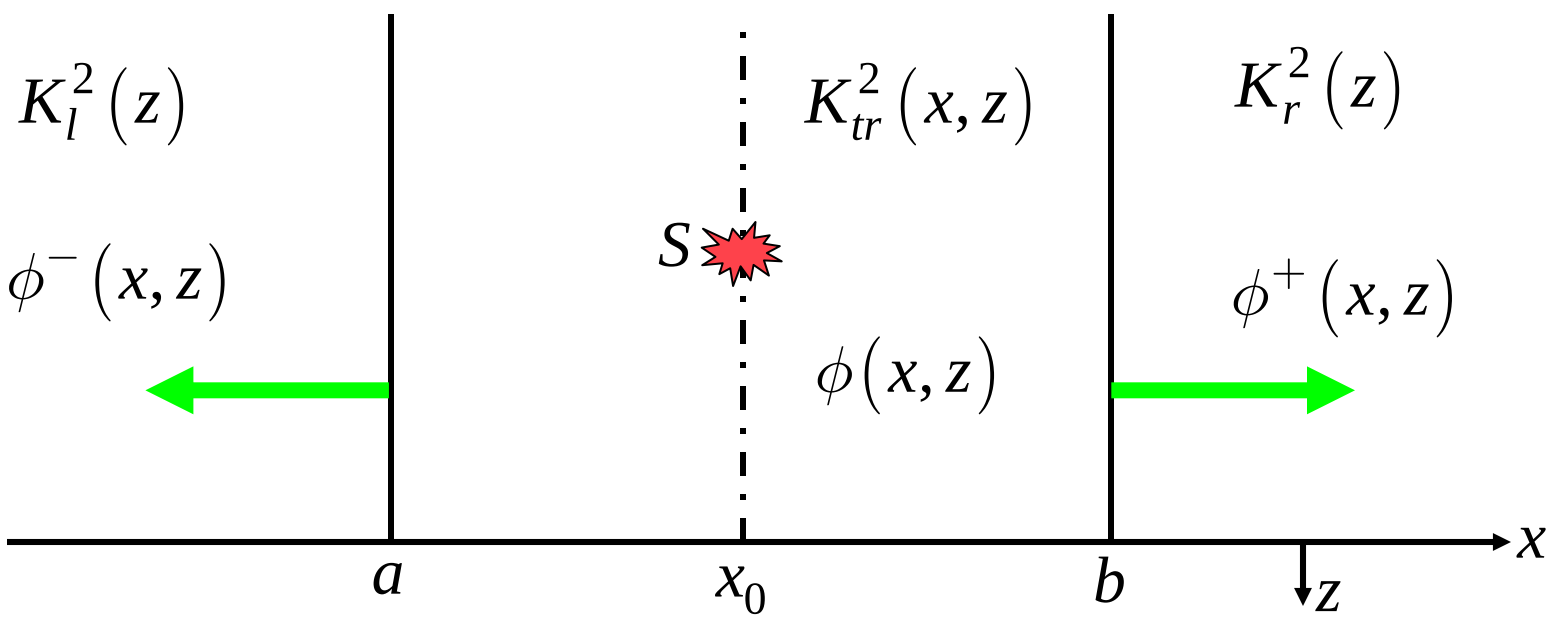
3) Extension of Fourier analysis to inhomogeneous environments

Mathematical Illustration

Scalar Helmholtz Equation

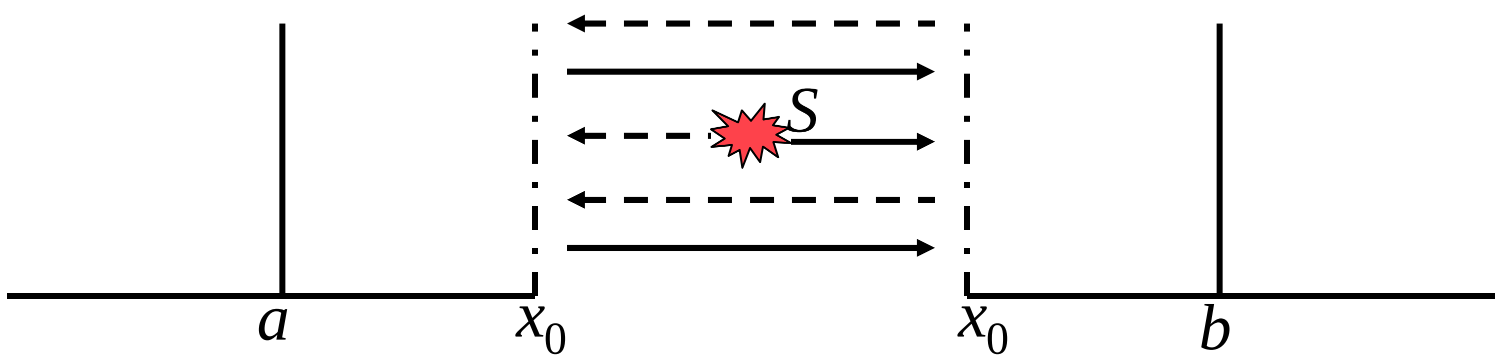
$$\left(\partial_x^2 + \underbrace{\partial_z^2 + \bar{k}^2 K^2(x,z)}_{\bar{k}^2 \mathbf{B}^2}\right)\phi(x,z) = -\delta(x - x_s)\delta(z - z_s)$$

General Radiation Formulation



Basic Scattering Picture

(1) Scattering Block Decomposition



(a) Individual block scattering problems

(b) "Glue" solutions together (block multiple scattering)

Basic Scattering Picture

(2) Fundamental Scattering Problem

Desire for one-way methods – simplest one-way marching scheme

$$\partial_x \begin{pmatrix} \phi \\ \partial_x \phi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\left(\partial_z^2 + \bar{k}^2 K^2(x,z)\right) & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \partial_x \phi \end{pmatrix}$$

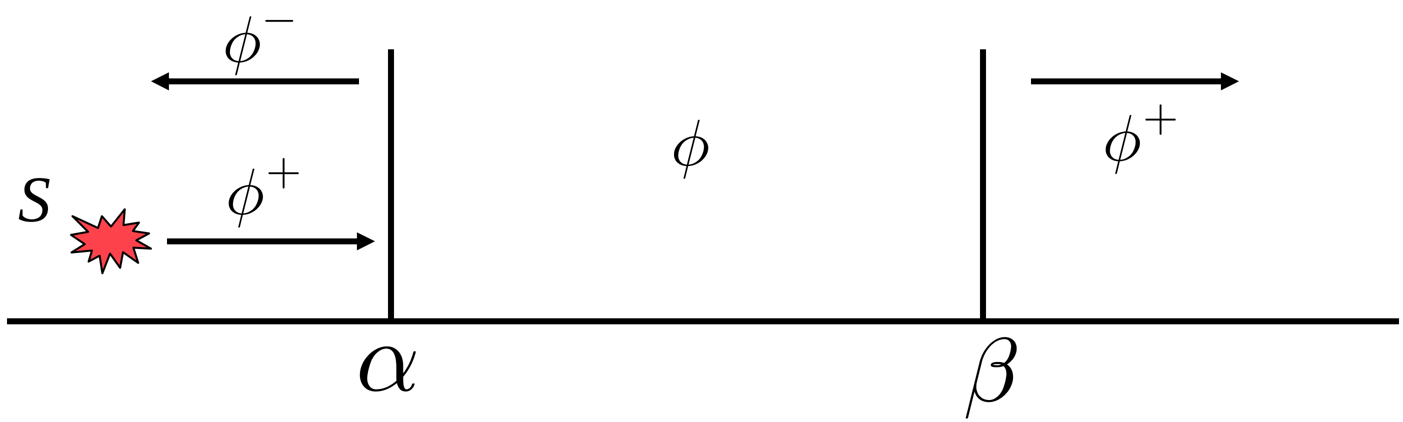
Ill-posed for simultaneous marching of wavefield and normal derivative in range direction

Non-independent initial data

Relationship between wavefield and normal derivative is key to well-posed marching method

Basic Scattering Picture

(3) Correct Scattering Kinematics (Geometry)



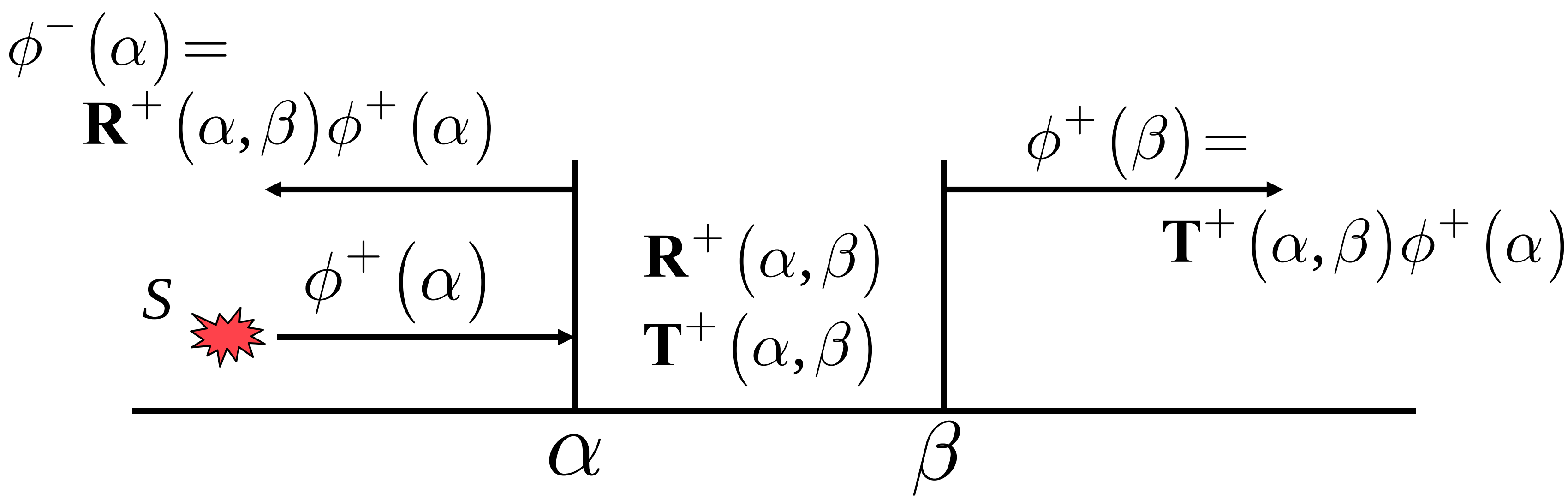
$$\phi^{\pm}(x,z) = \frac{1}{2}\left[\phi(x,z) \mp \left(i/\bar{k}\right)\mathbf{B}^{-1}\partial_x\phi(x,z)\right]$$

$$\mathbf{B} = \left(K^2(x,z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$$

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

Scattering Picture



Scattering operators associated with global block.

All internal multiple scattering incorporated into scattering operators.

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

Given $\phi(x_0, z)$, then propagation from x_0 is given by

$$\left(\left(1/\bar{k}\right)\partial_x + \mathbf{\Lambda}^+(x,b)\right)\phi(x,z) = 0$$

$$\left(\left(1/\bar{k}\right)\partial_x - \mathbf{\Lambda}^-(a,x)\right)\phi(x,z) = 0$$

where

$$\left(1/\bar{k}\right)\partial_x \mathbf{\Lambda}^+(x,b) = \left(\mathbf{\Lambda}^+(x,b)\right)^2 + \mathbf{B}^2(x)$$

with the initial condition

$$\mathbf{\Lambda}^+(b,b) = -i\mathbf{B}(b)$$

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

and

$$-\left(1/\bar{k}\right)\partial_x \mathbf{\Lambda}^-(a,x) = \left(\mathbf{\Lambda}^-(a,x)\right)^2 + \mathbf{B}^2(x)$$

with the initial condition

$$\mathbf{\Lambda}^-(a,a) = -i\mathbf{B}(a)$$

with

$$\mathbf{B}(x) = \left(K^2(x,z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$$

and where the initial wavefield is given by

$$\phi(x_0,z) = \left(1/\bar{k}\right)\left(\mathbf{\Lambda}^+(x_0,b) + \mathbf{\Lambda}^-(a,x_0)\right)^{-1}\delta(z - z_s)$$

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

(1) Now have two, well-posed marching problems, in opposite directions, done in succession.

(2) Must cover both directions for the two-way "elliptic wave propagation" problem: one direction to get the DtN operator and the other direction to propagate the total wavefield with the DtN operator.

(3) The idea is not to do the first marching procedure (DtN operator construction) computationally, but, rather, to solve the problem asymptotically. Thus, there will be only one, one-way marching computational procedure.

(4) The initial field calculation will also be done asymptotically.

Two Complementary Approaches

DtN Operator One-Way Reformulation

(1) Propagation operator:

$$\mathbf{\Lambda}^{+}\left(x,b\right)$$

(2) One-way wave equation:

$$\left(\left(1/\bar{k}\right)\partial_{x}+\mathbf{\Lambda}^{+}\left(x,b\right)\right)\phi\left(x,z\right)=0$$

(3) Fundamental solution (propagator):

$$\mathbf{G}^{+}\left(x,x_{0}\right)=\lim_{\delta\to 0}\left[\exp\left(-\bar{k}\,\delta\mathbf{\Lambda}^{+}\left(x_{N},b\right)\right)...\exp\left(-\bar{k}\,\delta\mathbf{\Lambda}^{+}\left(x_{1},b\right)\right)\right]$$

(4) Single-sweep algorithm on total wavefield

Seismo-Acoustic Operator Constructions

Propagation Operators

(1) Formal operator series/rational approximations on square-root operator (Claerbout, C. J. Thomson, Tappert, etc.)

$$\mathbf{B} = \left(\mathbf{I} + \mathbf{L}\right)^{1/2} \approx \mathbf{I} + \left(1/2\right)\mathbf{L} + \ldots$$

$$\mathbf{B} \approx \mathbf{B}^{\mathbf{ara}} = \mathbf{I} + \sum_{j=1}^{N} c_{j,N} \left(\mathbf{I} - \left(\mathbf{I} + b_{j,N}\mathbf{L}\right)^{-1}\right)$$

$$\mathbf{L} = \mathbf{B}^2 - \mathbf{I} = \left(\left(K^2\left(x,z\right) - 1\right)\mathbf{I} + \left(1/\bar{k}^2\right)\partial_z^2\right)$$

(2) Formal operator series on DtN operator (Bleistein, Zhang, and Zhang; Chapman)

Seismo-Acoustic Operator Constructions

Fundamental Solutions (Propagators)

(1) Operator rational approximations for the propagator (Collins)

$$\mathbf{G}^{+}(x,0) = \exp\left(i\bar{k}x\mathbf{B}\right)$$

(2) Path integral representations: time-domain wave equation (Schlottmann); fixed-frequency, one-way, anisotropic, elastic wave equation (C. J. Thomson)

Critical Comments

(1) $\left(K^2(x,z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$ does not define the operator

(2) Formal operator Taylor series do not define the operator

(3) Operator series expansions are nonuniform and singular

Phase Space and Path Integral Methods

General Problem:

At the simplest level, explicitly construct operator functions of the type:

$$\mathbf{B} = \left(K^2(z) + \left(1/\bar{k}^2 \right) \partial_z^2 \right)^{1/2}$$

and the corresponding fundamental solution,

$$\exp\left(i\bar{k}\, x\, \mathbf{B} \right)$$

History:

(1) Development of Quantum Mechanics

(2) Development of Modern Mathematical Asymptotics

Mathematical Framework

Homogeneous Half-Space $K^2(z) = K_0^2$

(1) Wave Equation

$$\left((i/\bar{k})\partial_x + \left(K_0^2 + (1/\bar{k}^2)\partial_z^2 \right)^{1/2} \right) \phi^+(x,z) = 0$$

$$(i/\bar{k})\partial_x \phi^+(x,z) + \frac{\bar{k}}{2\pi} \int_{\mathbb{R}^2} dp\, dz' \left(K_0^2 - p^2 \right)^{1/2}$$

$$\bullet \exp\left(i\bar{k}p(z - z') \right) \phi^+(x,z') = 0$$

Mathematical Framework

(2) Path Integral

$$G^{+}\left(x,z\middle|0,z'\right)=\lim_{N\to\infty}\int_{\mathbb{R}^{2N-1}}\prod_{j=1}^{N-1}dz_{j}\prod_{j=1}^{N}\left(\frac{\bar{k}}{2\pi}\right)dp_{j}$$

$$\bullet\exp\left[i\bar{k}\sum_{j=1}^{N}\left(p_{j}\left(z_{j}-z_{j-1}\right)+\left(\frac{x}{N}\right)\left(K_{0}^{2}-p_{j}^{2}\right)^{1/2}\right)\right]$$

Mathematical Framework

(3) Marching Numerical Algorithm

$$\phi^{+}\left(x+\Delta x, z\right)=$$

$$\int_{\mathbb{R}} dp \exp\left(i\bar{k}pz\right)\left[\exp\left(i\bar{k}\Delta x\left(K_0^2-p^2\right)^{1/2}\right)\hat{\phi}^{+}\left(x,p\right)\right]$$

$$=F^{-1}\left[\exp\left(i\bar{k}\Delta x\left(K_0^2-p^2\right)^{1/2}\right)F\left[\phi^{+}\left(x,z'\right),p\right],z\right]$$

Mathematical Framework

Transversely-Inhomogeneous Half-Space $K^2(x,z) = K^2(z)$

(1) Wave Equation

$$\left(\left(i/\bar{k}\right)\partial_x + \left(K^2(z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}\right)\phi^+(x,z) = 0$$

$$\left(i/\bar{k}\right)\partial_x\phi^+(x,z) + \frac{\bar{k}}{2\pi}\int_{\mathbb{R}^2} dp\,dz'\,\Omega_{\mathbf{B}}\left(p,\left(z+z'\right)/2\right)$$

$$\bullet \exp\left(i\bar{k}p\left(z-z'\right)\right)\phi^+(x,z') = 0$$

Mathematical Framework

(2) Path Integral

$$G^{+}\left(x, z \middle| 0, z'\right) = \lim_{N\to\infty} \int_{\mathbb{R}^{2N-1}} \prod_{j=1}^{N-1} dz_j \prod_{j=1}^{N} \left(\frac{\bar{k}}{2\pi}\right) dp_j$$

$$\bullet \exp\left[i\bar{k} \sum_{j=1}^{N} \left(p_j \left(z_j - z_{j-1}\right) + \left(\frac{x}{N}\right) h_{\mathbf{B}}^{s}\left(p_j, z_j\right) \right) \right]$$

Mathematical Framework

(3) Marching Numerical Algorithm

$$\phi^{+}\left(x+\Delta x, z\right)\approx$$

$$\int_{\mathbb{R}} dp \exp\left(i\overline{k}pz\right)\left[\exp\left(i\overline{k}\,\Delta x h_{\mathbf{B}}^{s}\left(p,z\right)\right)\hat{\phi}^{+}\left(x,p\right)\right]$$

$$h_{\mathbf{B}}^{s}\left(p,q\right)=\left(\overline{k}/\pi\right)\int_{\mathbb{R}^2} dsdt\Omega_{\mathbf{B}}\left(s,t\right)\exp\left(-2i\overline{k}\left(q-t\right)\left(p-s\right)\right)$$

Mathematical Framework

Arbitrary transverse inhomogeneity

Operator symbols determined by appropriate composition equations, e.g.,

$$\Omega_{\mathbf{B}^2}\left(p,q\right)=K^2\left(q\right)-p^2=$$

$$\left(\bar{k}/\pi\right)^2\int_{\mathbb{R}^4}dtdsdvdu\Omega_{\mathbf{B}}\left(t+p,s+q\right)$$

$$\bullet\,\Omega_{\mathbf{B}}\left(v+p,u+q\right)\exp\left(2i\bar{k}\left(sv-tu\right)\right)$$

supplemented with right-traveling-wave radiation condition

Mathematical Framework

Natural Question

Can we take the desired exact and uniform approximate operator symbol constructions directly from the quantum mechanical and modern mathematical asymptotic literatures?

Answer – No!

Why?

(1) Mathematical analysis → Quantum mechanical results

(2) While the mathematical analysis provides the complete framework for the equations, microlocal analysis (asymptotics) only considers part of the solution – it is an approximation

Mathematical Framework

Why?

(3) Approximation appropriate for time-domain formulations not frequency-domain Helmholtz equation (propagation of singularities versus smoothing)

(4) Will result in nonuniform, singular approximations for Helmholtz equation

Illustration – Weyl composition equation

$$\Omega_{\mathbf{B}^2}\left(p,q\right)=K^2\left(q\right)-p^2=\left(\bar{k}/\pi\right)^2\int_{\mathbb{R}^4} dtdsdvdu\,\Omega_{\mathbf{B}}\left(t+p,s+q\right)$$

$$\bullet\Omega_{\mathbf{B}}\left(v+p,u+q\right)\exp\left(2i\bar{k}\left(sv-tu\right)\right)$$

Mathematical Framework

Illustration – Weyl composition equation

In the high-frequency $\left(\bar{k} \to \infty\right)$ limit, this takes the form

$$\Omega_{\mathbf{B}^2}\left(p,q\right) = K^2\left(q\right) - p^2 = \lim_{\substack{\eta \to p \\ y \to q}} \cos\left(\frac{1}{2\bar{k}}\left(\partial_\eta \partial_q - \partial_p \partial_y\right)\right) \Omega_{\mathbf{B}}\left(p,q\right) \Omega_{\mathbf{B}}\left(\eta,y\right)$$

Substituting the expansion

$$\Omega_{\mathbf{B}}\left(p,q\right) \sim \Omega_{\mathbf{B}}^{(0)}\left(p,q\right) + \left(1/\bar{k}^2\right) \Omega_{\mathbf{B}}^{(2)}\left(p,q\right) + \ldots$$

results in

$$\Omega_{\mathbf{B}}\left(p,q\right) \sim \left(K^2\left(q\right) - p^2\right)^{1/2} - \frac{K^3\left(q\right) \partial_q^2 K\left(q\right)}{8\bar{k}^2 \left(K^2\left(q\right) - p^2\right)^{5/2}} + \ldots$$

Mathematical Framework

Illustration – Weyl composition equation

The corresponding result in the standard calculus is

$$h_{\mathbf{B}}^{s}(p,q) \sim \left(K^2(q) - p^2\right)^{1/2} + \frac{K(q)\partial_q K(q)\, p}{2i\bar{k}\left(K^2(q) - p^2\right)^{3/2}} + O\left(1/\bar{k}^2\right)$$

For the case of the DtN operator symbol, the result is

$$h_{\mathbf{\Lambda}^+}^{s}(x,p,q) \sim -i\left(K^2(x,q) - p^2\right)^{1/2} + \frac{K(x,q)\partial_x K(x,q)}{2\bar{k}\left(K^2(x,q) - p^2\right)} + O\left(1/\bar{k}^2\right)$$

All are nonuniform, singular expansions.

Proper theory requires going beyond examples from quantum mechanics and results from modern mathematical asymptotics.

Exact Symbols – Focusing Quadratic

Profile

$$K^2(q) = K_0^2 - \omega^2 q^2, n = 2, K_0, \omega > 0, q \varepsilon \mathbb{R}$$

Weyl operator symbol

$$\Omega_{\mathbf{B}}(p,q) = -\exp(i\pi/4)\left(\frac{\varepsilon}{2}\right)^{1/2}\left(\frac{1}{\pi}\right)\int_L d\tau \varsigma\left(1/2, (-i/2\pi)\tau, \exp(2i\pi Y)\right)$$

$$\bullet \exp\left(Y\tau - \mathrm{X}\tanh\tau\right)\operatorname{sec}h\tau\left(Y - \mathrm{X}\operatorname{sec}h^2\tau - \tanh\tau\right), Y \neq 0,1,2,\ldots$$

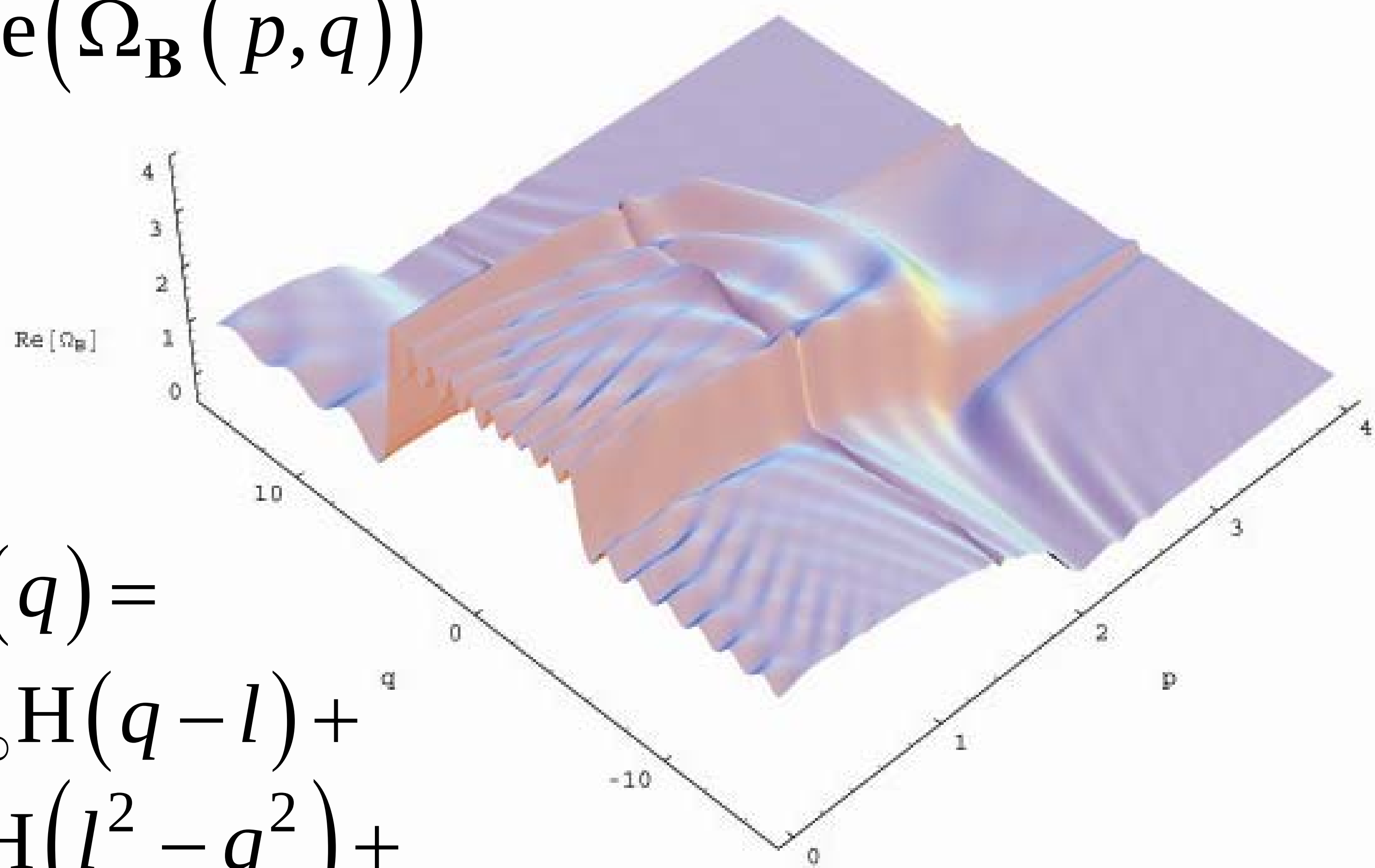
$$\mathrm{X} = (1/\varepsilon)\left(\omega^2 q^2 + p^2\right), Y = K_0^2/\varepsilon, \varepsilon = \omega/\overline{k}$$

$$\varsigma(\sigma,\Delta,\xi) = \sum_{n=0}^{\infty}\frac{\xi^n}{(n+\Delta)^{\sigma}}, \Delta \neq 0,-1,-2,\ldots,|\xi| < 1, + \text{ anal. conts.}$$

(Lerch transcendental function)

Exact Operator Symbol
3-Layer Profile

$$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$

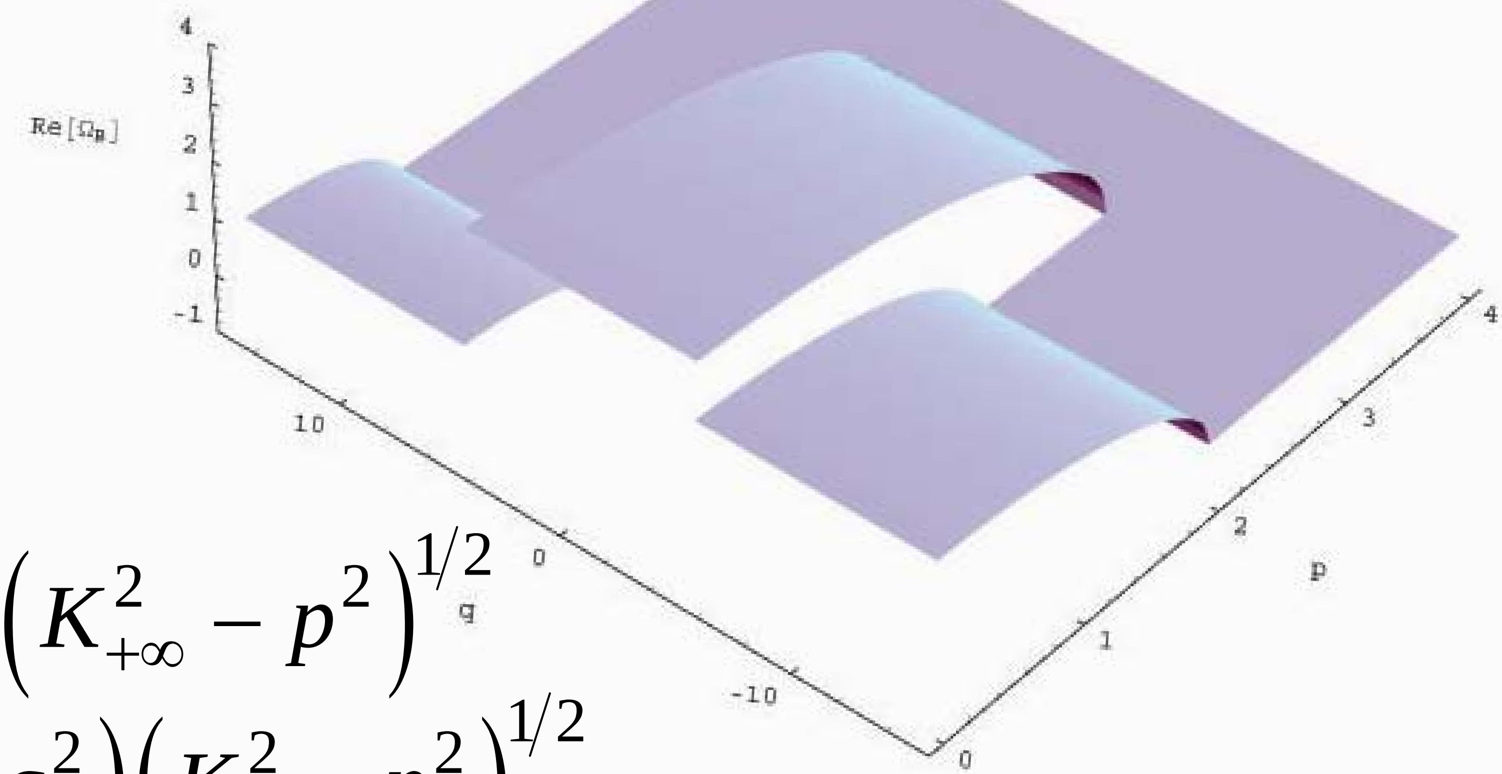


$$K^{2}\left(q\right)=$$
$$K_{+\infty}^{2}\mathrm{H}\left(q-l\right)+$$
$$K_{1}^{2}\mathrm{H}\left(l^{2}-q^{2}\right)+$$
$$K_{-\infty}^{2}\mathrm{H}\left(-q-l\right)$$

$$K_{-\infty}=2, K_{1}=3, K_{+\infty}=1$$
$$\bar{k}=1, l=5$$

Locally Homogeneous Approximation
3-Layer Profile

$$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$

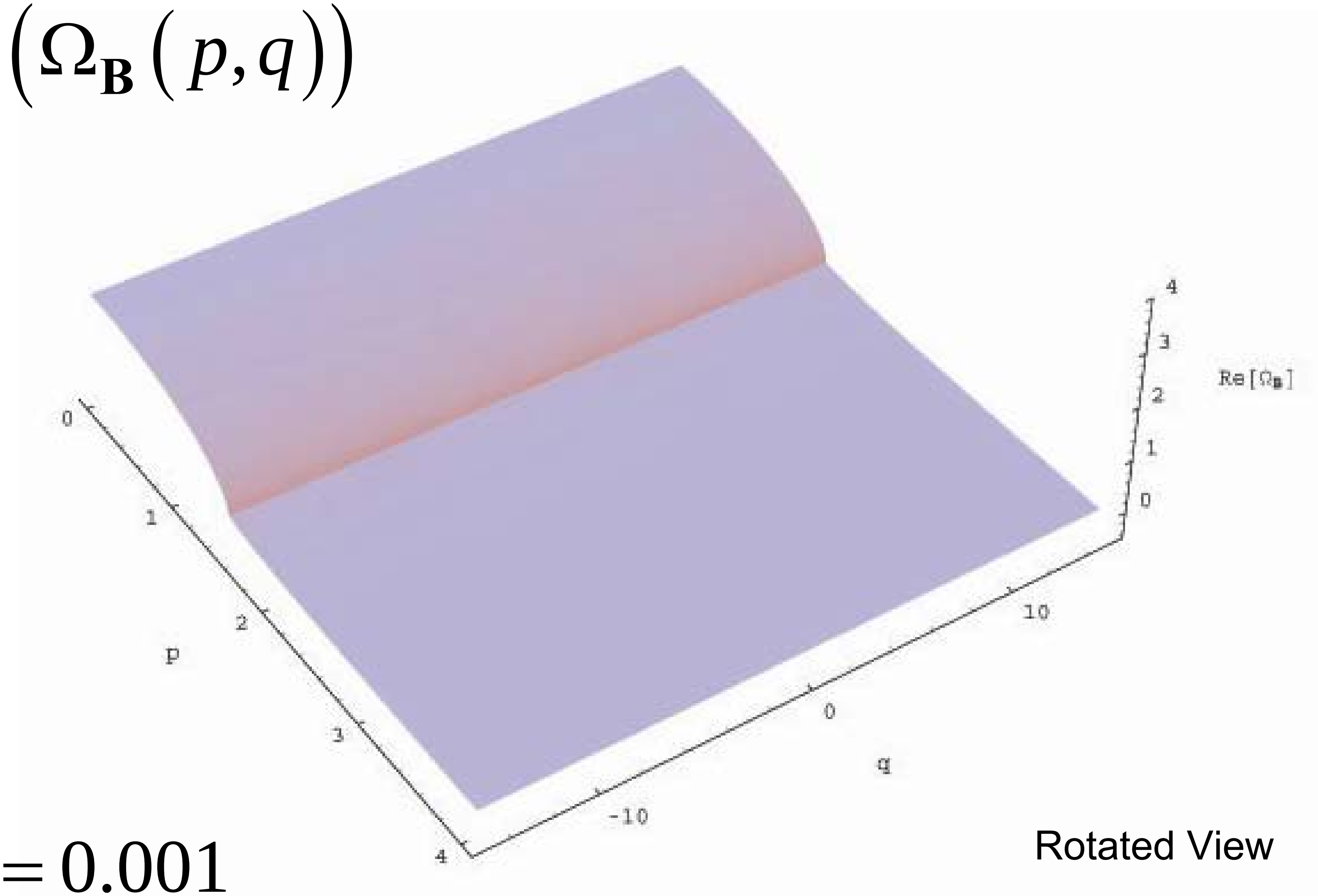


$$\Omega_{\mathbf{B}} =$$
$$\mathrm{H}\left(q-l\right)\left(K_{+\infty}^{2}-p^{2}\right)^{1/2}$$
$$+\mathrm{H}\left(l^{2}-q^{2}\right)\left(K_{1}^{2}-p^{2}\right)^{1/2}$$
$$+\mathrm{H}\left(-q-l\right)\left(K_{-\infty}^{2}-p^{2}\right)^{1/2}$$

$$K_{-\infty}=2, K_{1}=3, K_{+\infty}=1$$
$$l=5$$

Exact Operator Symbol
3-Layer Profile

$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



$\bar{k} = 0.001$

Rotated View

Uniform High-Frequency Expansion

Operator Symbol

(1) n = 2

(2) Refractive index field varies “slowly” on wavelength scale

$$\Omega_{\mathbf{B}}\left(p,q\right)\sim\left(K^{2}\left(q\right)-p^{2}\right)^{1/2}+\int_{0}^{\infty}du\cos\left(\bar{k}pu\right)$$

$$\bullet\left(K^{2}\left(q\right)\left(A\left(\frac{H_{1}^{(1)}\left(\bar{k}I_{0}\right)}{I_{0}}+\frac{C}{\bar{k}}H_{0}^{(1)}\left(\bar{k}I_{0}\right)\right)-\frac{H_{1}^{(1)}\left(\bar{k}K\left(q\right)u\right)}{K\left(q\right)u}\right)\right)$$

where

$$A=\left(\frac{I_{0}}{I_{1}}\right)^{3/2}\left(\frac{1}{K^{2}\left(q\right)}\left(K\left(q+u/2\right)K\left(q-u/2\right)\right)^{-1/2}\right)$$

Uniform High-Frequency Expansion

Operator Symbol

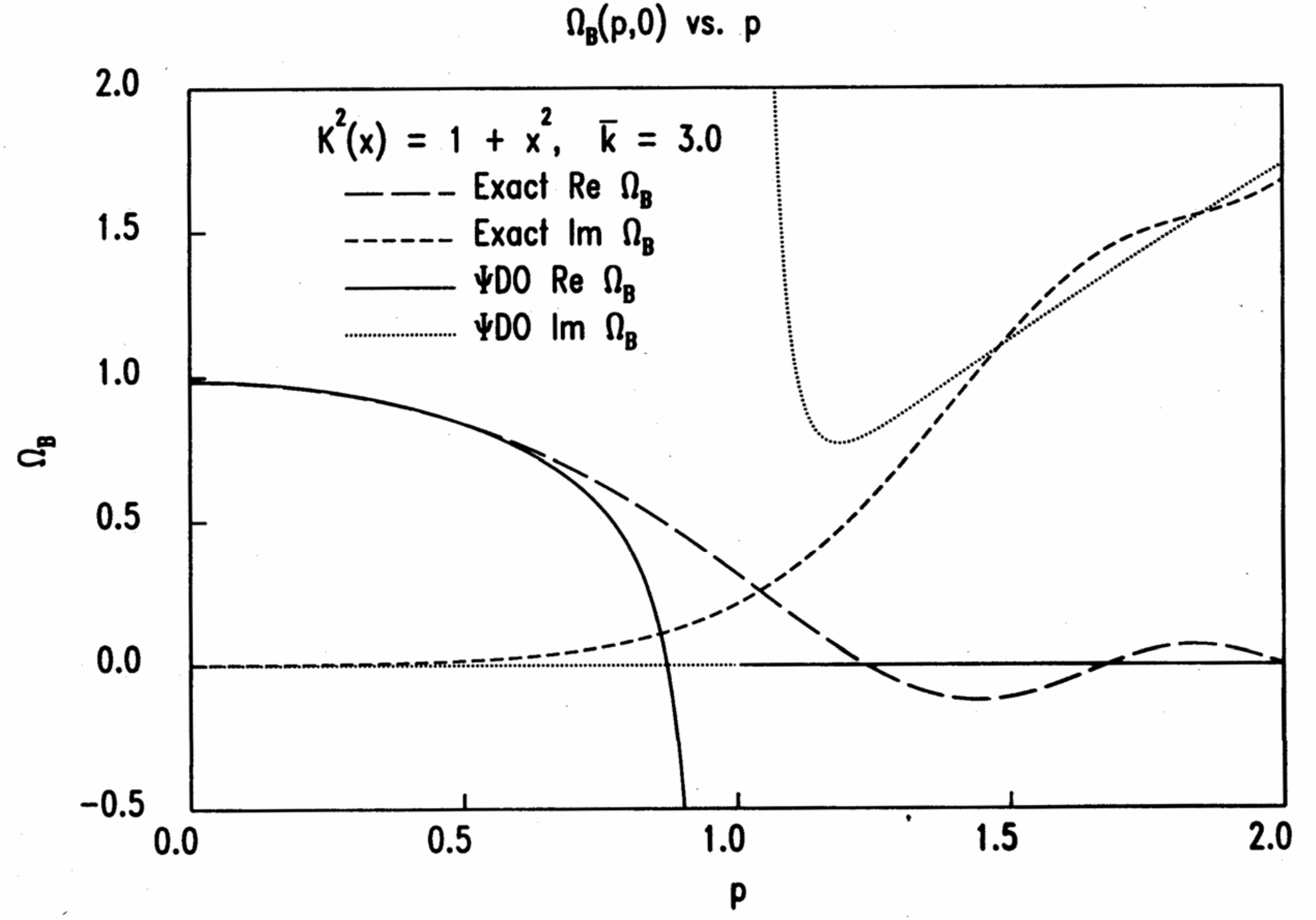
$$C=\left(\frac{1}{8I_0}\right)\left(\frac{15I_2}{I_1^2}-\frac{3}{I_0}-\frac{6}{I_1}\left(\frac{1}{K^2\left(q+u/2\right)}+\frac{1}{K^2\left(q-u/2\right)}\right)\right.$$

$$\left.+2\left(\frac{K'\left(q-u/2\right)}{K^2\left(q-u/2\right)}-\frac{K'\left(q+u/2\right)}{K^2\left(q+u/2\right)}\right)-\tilde{I}\right)$$

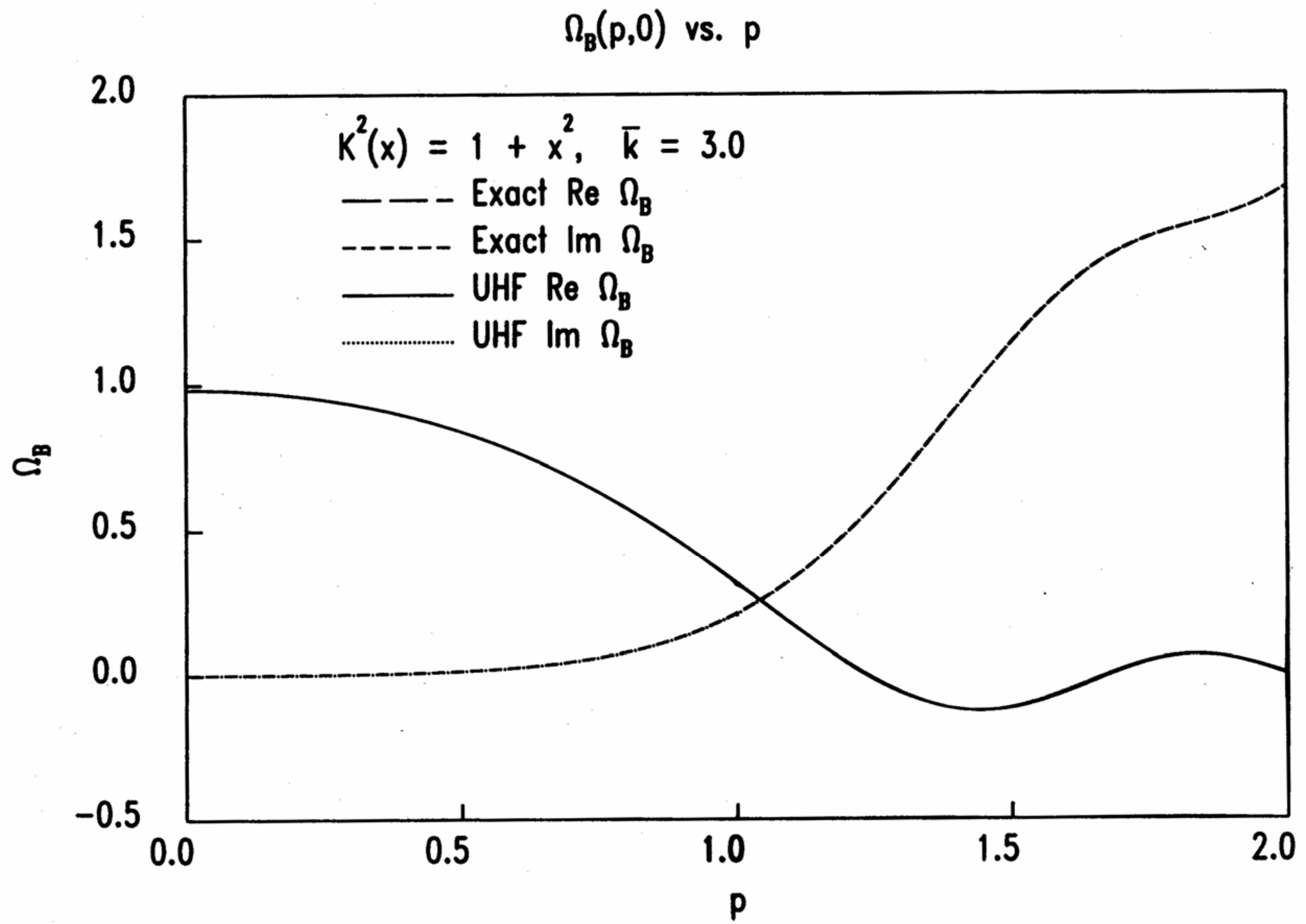
$$I_m=\int_{q-u/2}^{q+u/2}dt\left(K\left(t\right)\right)^{1-2m},m=0,1,2$$

$$\tilde{I}=\int_{q-u/2}^{q+u/2}dt\left(\frac{\left(K'\left(t\right)\right)^2}{K^3\left(t\right)}\right)$$

Pseudodifferential Versus Exact



UHF Versus Exact



Uniform High-Frequency Wave Theory

Path Integral + UHF Standard Operator Symbol

(1) Distinct from high-frequency approximations made directly on the wavefield, including the globally uniform, high-frequency constructions of Maslov (Fourier transform) and Klauder (coherent-state transform)

(2) Incorporates wave (diffraction) effects via “sum over paths” in phase space and uniform approximation of phase functional (operator symbol) in path integral

(3) Correct incorporation of both high-propagating-angle and post-critical wave phenomena

(4) Essentially the difference of classical, high-frequency asymptotic wavefield as (1) global solution and (2) local solution repeatedly composed to produce a global solution

Applications to Seismic Imaging

(1) UHF symbol imaging (Born, decoupled wavefield extrapolation)

(a) Full symbol – accuracy assessment

(b) Simplified symbol approximation for increased computational speed

(2) DtN symbol imaging (Born, coupled wavefield extrapolation)

Mathematical Framework

Generally-Inhomogeneous Composition Equation

$$\left(1/\bar{k}\right)\partial_x \Omega_{\mathbf{\Lambda}^+}\left(x,b;p,q\right)=$$

$$\left(\bar{k}/\pi\right)^2\int_{\mathbb{R}^4} dtdsdvdu\Omega_{\mathbf{\Lambda}^+}\left(x,b;t+p,s+q\right)$$

$$\bullet\,\Omega_{\mathbf{\Lambda}^+}\left(x,b;v+p,u+q\right)\exp\left(2i\bar{k}\left(sv-tu\right)\right)$$

$$+K^2\left(x,q\right)-p^2$$

Initial condition: $\Omega_{\mathbf{\Lambda}^+}\left(b,b;p,q\right)=-i\Omega_{\mathbf{B}}\left(b;p,q\right)$

Generalized (nonlocal) Riccati equation

Applications to Seismic Imaging

(3) Direct, non-perturbative, one-way marching inversion algorithm

(a) Full multiple scatter

(b) Exact "imaging conditions"

$\left\{ \phi^{+}, \phi^{-} \right\}$ Through the Reflection operator

$\left\{ \phi, \partial_{x}\phi \right\}$ Through the DtN operator

Applications to Seismic Imaging

(c) Operator symbol as "data"

(d) Complementary asymptotic/numerical
downward continuation of the "data"

Potential Benefits

(1) Accommodation of primaries and multiples
Exact, well-posed, one-way reformulation
Corresponding exact "imaging condition"

(2) High-angle imaging in heterogeneous media

(3) Better amplitude estimation through improved flux conservation and increased accuracy

Conclusions

1) One-way wave equations can be constructed that incorporate all of the forward and backward scattering inherent in the two-way wave equations.

2) The above one-way formulations can be made explicit by exploiting the correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics.

3) Effectively, these constructions extend Fourier analysis to inhomogeneous environments.

4) Uniform high-frequency asymptotic operator symbol approximations extend GPSPI.

5) Everything just runs like the GPSPI algorithm.

6) Exact imaging conditions are constructed that account, in principle, for all of the multiple scattering.