Electromagnetic Imaging of Buried Objects

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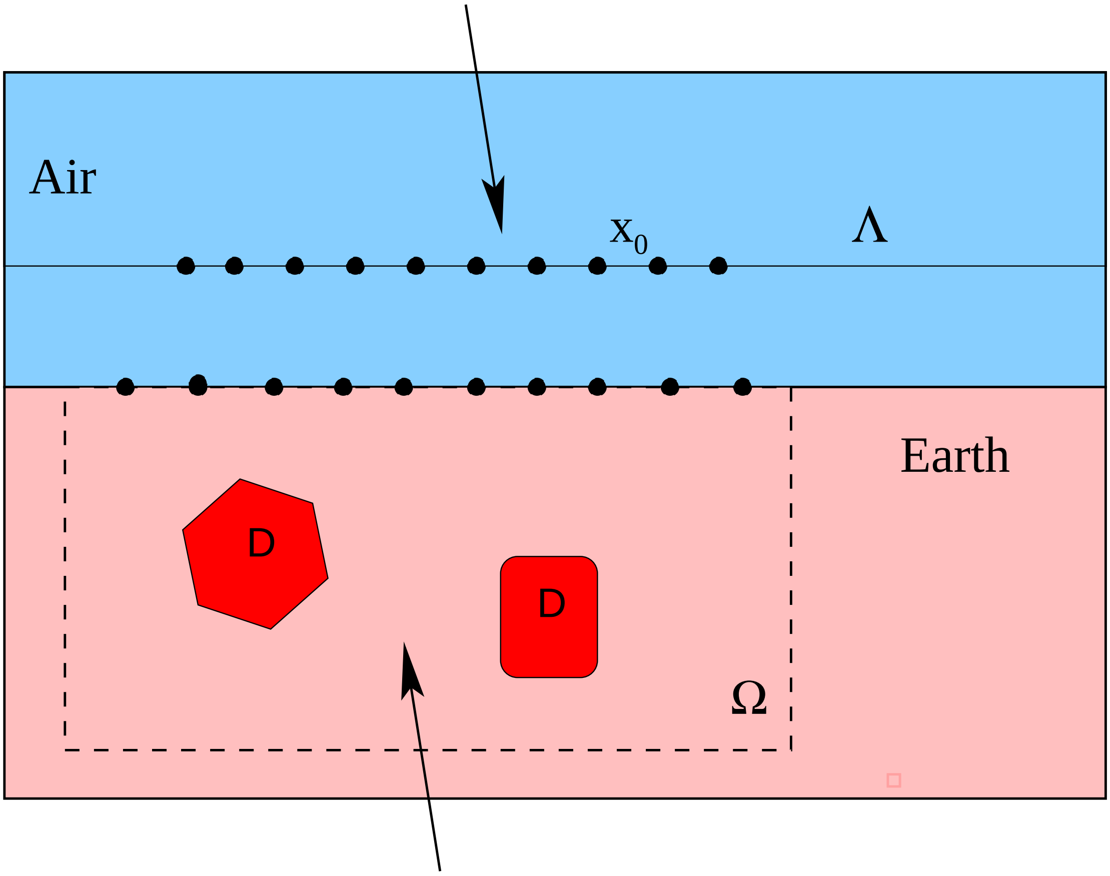
Detection of Buried Objects

$$\begin{cases} \nabla \times E - ikH &= 0 \\ \nabla \times H - ikE &= 0 \end{cases}$$



$$\begin{cases} \nabla \times E - ikH = 0 \\ \nabla \times H - ikn_b E = 0 \end{cases}$$

Detection of Buried Objects

- $E = E^i + E^s$.
- E^s decays appropriately as $|x| \to \infty$.
- E^i is an electric dipole with source at $x = x_0$.
- On ∂D either

1. $\nu \times E = 0$ on ∂D or
2. $[\nu \times E] = 0$ across $\partial D = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$

   $[\nu \times H] = 0$ across Γ_1

   $[\nu \times H] = \eta(x)(\nu \times E) \times \nu$ across Γ_2

   where η is the surface conductivity and in D the index of refraction is a matrix $N(x)$, i.e. D is anisotropic.

Inverse Scattering Problem

From a knowledge of

$$\nu \times E(x, x_0, p) \text{ and } \nu \times H(x, x_0, p)$$

for x on the surface of the earth, $x_0 \in \Lambda$ and two linearly independent polarizations p tangential to Λ at x_0, determine D.

Remark: Γ_2 a perfect conductor corresponds to $\eta = \infty$. In particular D a perfect conductor corresponds to $\Gamma_2 = \partial D$, $\Gamma_1 = \emptyset$, $\eta = \infty$.

Remark: Our methods are also applicable to the case when $\nu \times E$ and $\nu \times H$ are measured above the surface of the earth.

Reciprocity Gap Fuctional

Let $k_b^2 = k^2 n_b$ and

$$\mathbb{H} = \{W \in H(curl, \Omega) : \ \nabla \times \nabla \times W - k_b^2 W = 0\}.$$

If $E(x, x_0, p)$ and $H(x, x_0, p) = \frac{1}{ik_b} \nabla_x \times E(x, x_0, p)$ is the total field then the reciprocity gap functional $\mathcal{R}(E, \cdot) : \mathbb{H} \rightarrow L_t^2(\Lambda)$ is defined by

$$\mathcal{R}(E, W) =$$

$$\int_{\partial\Omega} [(\nu \times E) \cdot (\nabla \times W) - (\nu \times W) \cdot (\nabla \times E)] \, ds.$$

Instead of the whole space $\mathbb{H}$, it suffices to consider a dense subset.

Examples of Dense Subsets

- $W = E_g$ with E_g being the electric Herglotz wave function given by

$$E_g(x) = \int_{S^2} g(d) e^{ik_b d\cdot x} ds(d),$$

$g \in L^2_t(S^2)$.

In this case $\mathcal{R}(E,\, E_g) = \mathcal{R}g : g \in L^2_t(S^2) \to L^2_t(\Lambda)$.

- $W = A\phi$ with $A\phi$ being the single layer potential given by

$$(A\phi)(x) = \nabla \times \nabla \times \int_{\partial\Omega} \phi(y) \frac{e^{ik_b|x-y|}}{4\pi|x-y|} ds(y),$$

$\phi \in L^2_{div}(\partial\Omega)$.

In this case
$\mathcal{R}(E,\, A\phi) = \mathcal{R}\phi : \phi \in L^2_{div}(\partial\Omega) \to L^2_t(\Lambda)$.

Solving the Inverse Scattering Problem

Considering $A\phi$ for $\phi \in L^2_{div}(\partial\Omega)$, we look for a solution ϕ to the equation

$$\mathcal{R}\phi = \mathcal{R}(E,\ E_e(\cdot\,, z, q, k_b)) \qquad z \in \Omega$$

where $\mathcal{R}\phi = \mathcal{R}(E,\ A\phi)$ and

$$E_e(x; z, q, k_b) = \frac{i}{k_b}\nabla_x \times \nabla_x \times q\,\frac{e^{ik_b|x-z|}}{4\pi|x-z|}$$

$$H_e(x; z, q, k_b) = \nabla_x \times q\,\frac{e^{ik_b|x-z|}}{4\pi|x-z|}$$

is the electric dipole located at $z \in \Omega$.

Solving the Inverse Scattering Problem

Theorem: For every $\epsilon > 0$, there exists an approximate solution ϕ_ϵ^z satisfying

$$\|\mathcal{R}\phi_\epsilon^z - \mathcal{R}(E,\ E_e(\cdot\,;z,q,k_b))\|_{L^2(\Lambda)} < \epsilon \qquad z \in \Omega$$

that behaves as follows:

- For $z \in D$,

$$\lim_{\epsilon \to 0} \|A\phi_\epsilon^z\|_{H(D,curl)} < \infty.$$

- For each $\epsilon > 0$,

$$\lim_{z \to \partial D} \|A\phi_\epsilon^z\|_{H(D,curl)} = \infty$$

  and

$$\lim_{z \to \partial D} \|\phi_\epsilon^z\|_{L^2_{div}(\partial\Omega)} = \infty.$$

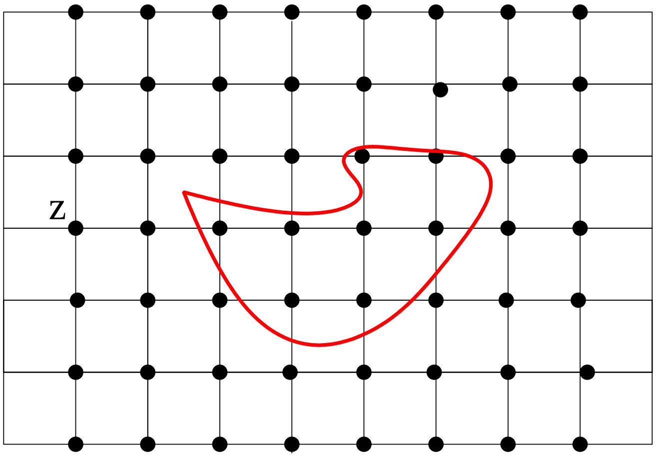
- For $z \in \Omega \setminus \overline{D}$

$$\lim_{\epsilon \to 0} \|A\phi_\epsilon^z\|_{H(D,curl)} = \infty$$

  and

$$\lim_{\epsilon \to 0} \|\phi_\epsilon^z\|_{L^2_{div}(\partial\Omega)} = \infty.$$

Numerical Implementation



- Construct a grid $\mathcal{G}$.
- For $z_i \in \mathcal{G}$, solve the regularized equation

$$(\alpha I + \mathcal{R}^* \mathcal{R}) \phi_{z_i,q} = F_{z_i,q}.$$

- Evaluate

$$\Phi(z_i) = \frac{1}{3} \left(\|\phi_{z_i,q_1}\|^{-1} + \|\phi_{z_i,q_2}\|^{-1} + \|\phi_{z_i,q_3}\|^{-1} \right)$$

for $z_i \in \mathcal{G}$ and three linearly independent vectors $q_1, q_2, q_3 \in \mathbb{R}^3$.

- Fix $C > 0$ and visualize the boundary by plotting

$$\Phi(z) = C \max_{z_i \in \mathcal{G}} \Phi(z_i).$$

A Special Case

Assume

- $n_b = 1.$
- $E^i = \dfrac{i}{k}\nabla \times \nabla \times p\, e^{ikx\cdot d}, \quad |d| = 1.$

Then

$$E^s(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}$$

as $|x| \to \infty$ where $\hat{x} = x/|x|$. The inverse scattering problem is now to determine D from a knowledge of the far field pattern

$$E_\infty(\hat{x}) = E_\infty(\hat{x}, d, p)$$

for $\hat{x} \in \Omega := \{x : |x| = 1\}$ and two linearly independent polarizations p tangential to Ω.

A Special Case

We define the far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_{\Omega} E_{\infty}(\hat{x}, d, g(d))\, ds(d).$$

Given $g \in L^2_t(\Omega)$, Fg is the far field pattern of the scattered field corresponding to the incident field being a Herglotz wave function with kernel g.

Now consider the far field equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)$$

where $E_{e,\infty}$ is the electric far field pattern of an electric dipole with source point z and polarization q.
For g_z^{ϵ} an approximate solution to the far field equation, ∂D is determined by

$$\lim_{\substack{z \to \partial D \\ z \in D}} \|g_z^{\epsilon}\|_{L^2_t(\Omega)} = \infty.$$

Information on η

Let g_z^ϵ be an approximate solution of the far field equation

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)$$

and $E_{g_z^\epsilon}$ the Herglotz wave function with kernel g_z^ϵ.

Assume η is a constant. Then we have the following theorem:

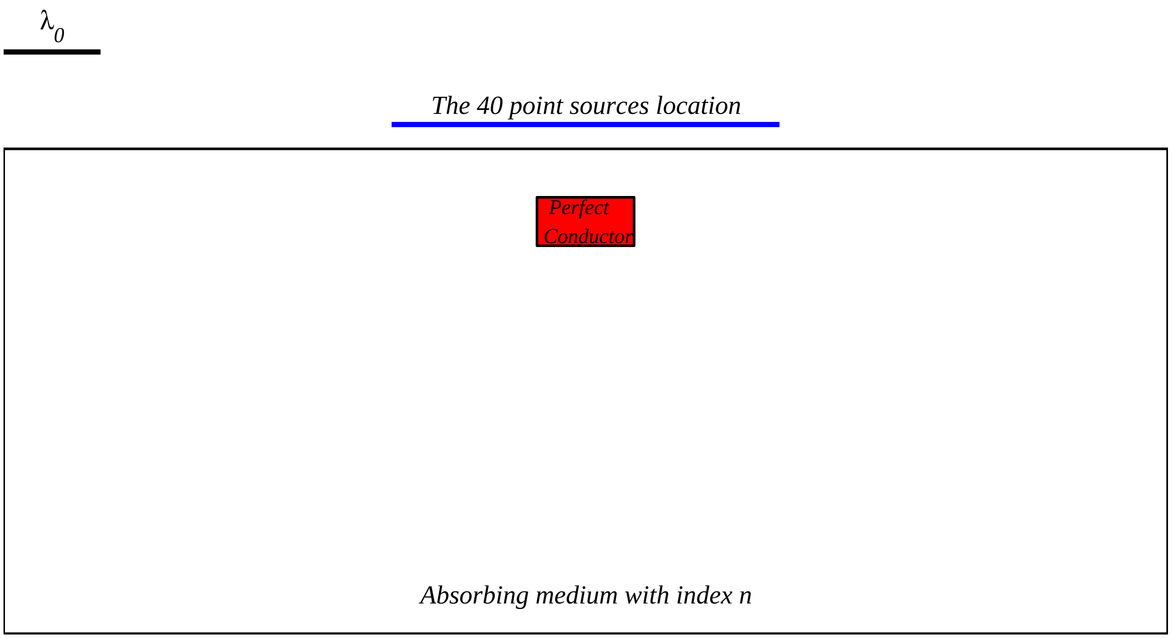
Theorem *(Cakoni-Colton-Monk)*:
Let z be any arbitrary point in D and q a vector in $\mathbb{R}^3$. Then

$$\eta \approx \frac{-\frac{k^2}{6\pi}\|q\|^2 + \Re\left(q \cdot E_{g_z^\epsilon}(z)\right)}{\|\nu \times \left(E_{g_z^\epsilon}(\,\cdot\,) + E_e(\cdot\,, z, q)\right)\|^2_{L_t^2(\Gamma_2)}}.$$

Corollary: For $z \in D$, $q \in \mathbb{R}^3$, we have that for any $\epsilon > 0$

$$\eta \geq \frac{-\frac{k^2}{6\pi}\|q\|^2 + \Re\left(q \cdot E_{g_z^\epsilon}(z)\right)}{\|E_{g_z^\epsilon}(\,\cdot\,) + E_e(\cdot\,, z, q)\|^2_{L^2(\partial D)}} + O(\epsilon).$$

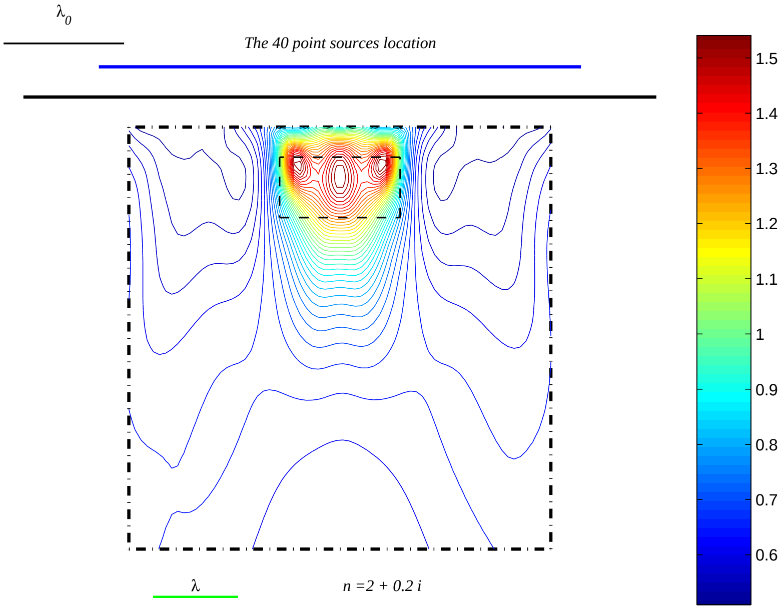
Numerical Examples ($\mathbb{R}^2$)



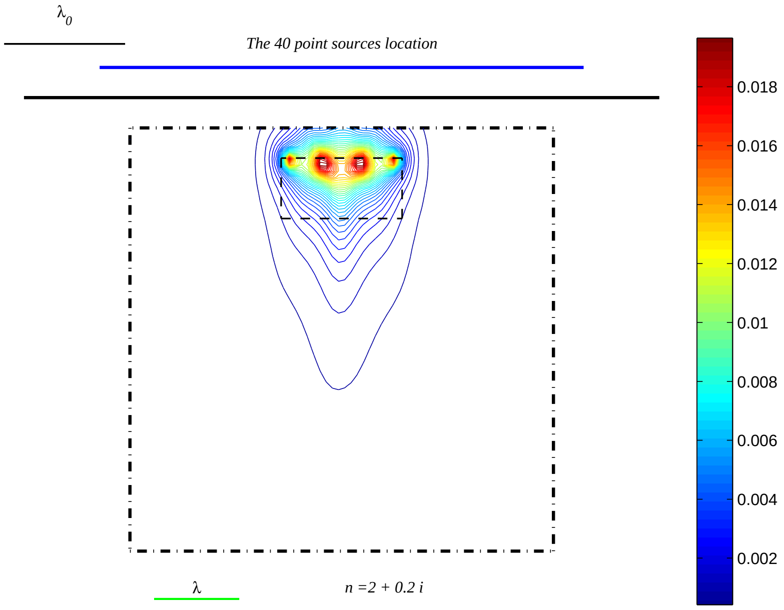
Sources are located on the blue line.
Measurements are taken on the interface

We will reconstruct D using the reciprocity gap functional where due to the absorption we can ignore the total field on the base and sides.

Numerical Examples ($\mathbb{R}^2$)



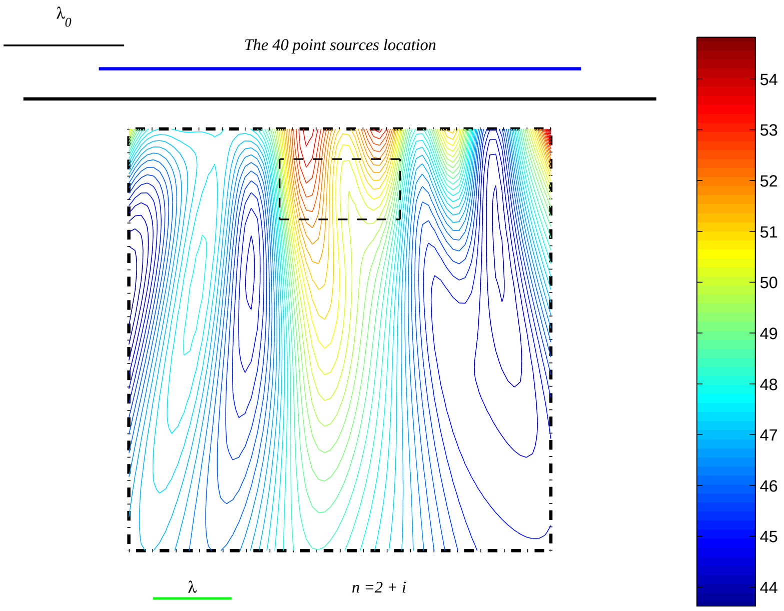
Using Herglotz functions



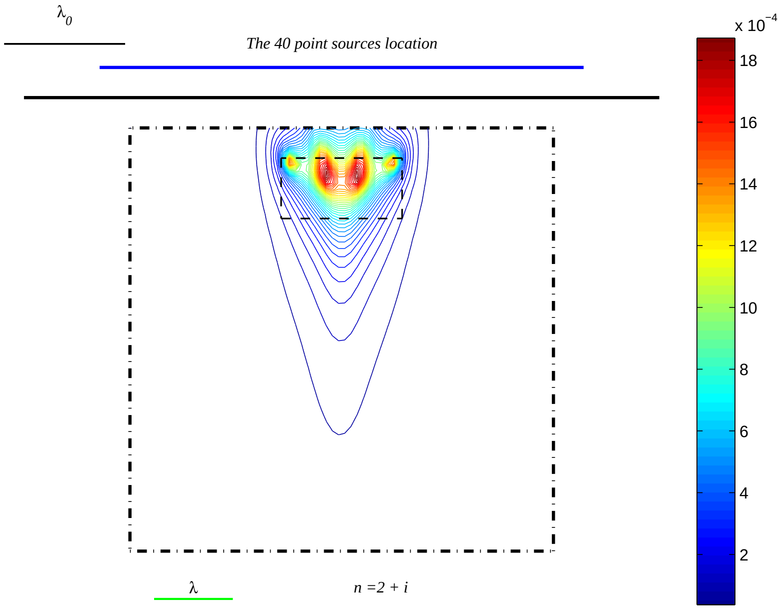
Using single layer potentials

Reconstruction for $n_b = 2 + 0.5i$

Numerical Examples ($\mathbb{R}^2$)



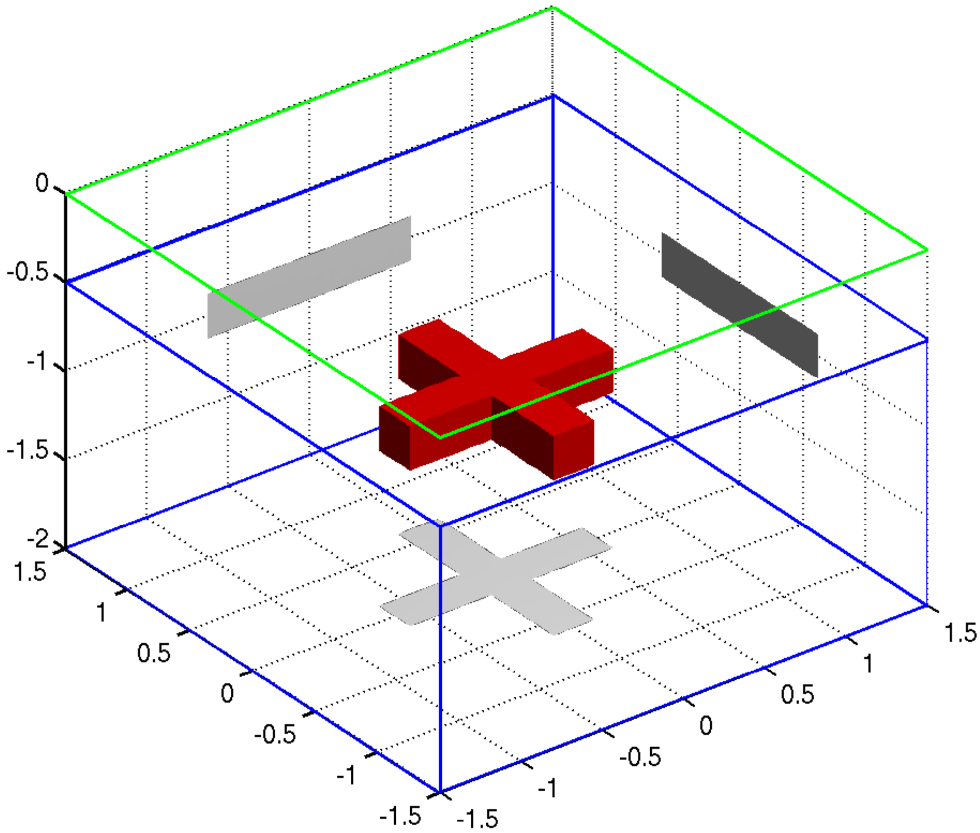
Using Herglotz functions



Using single layer potentials

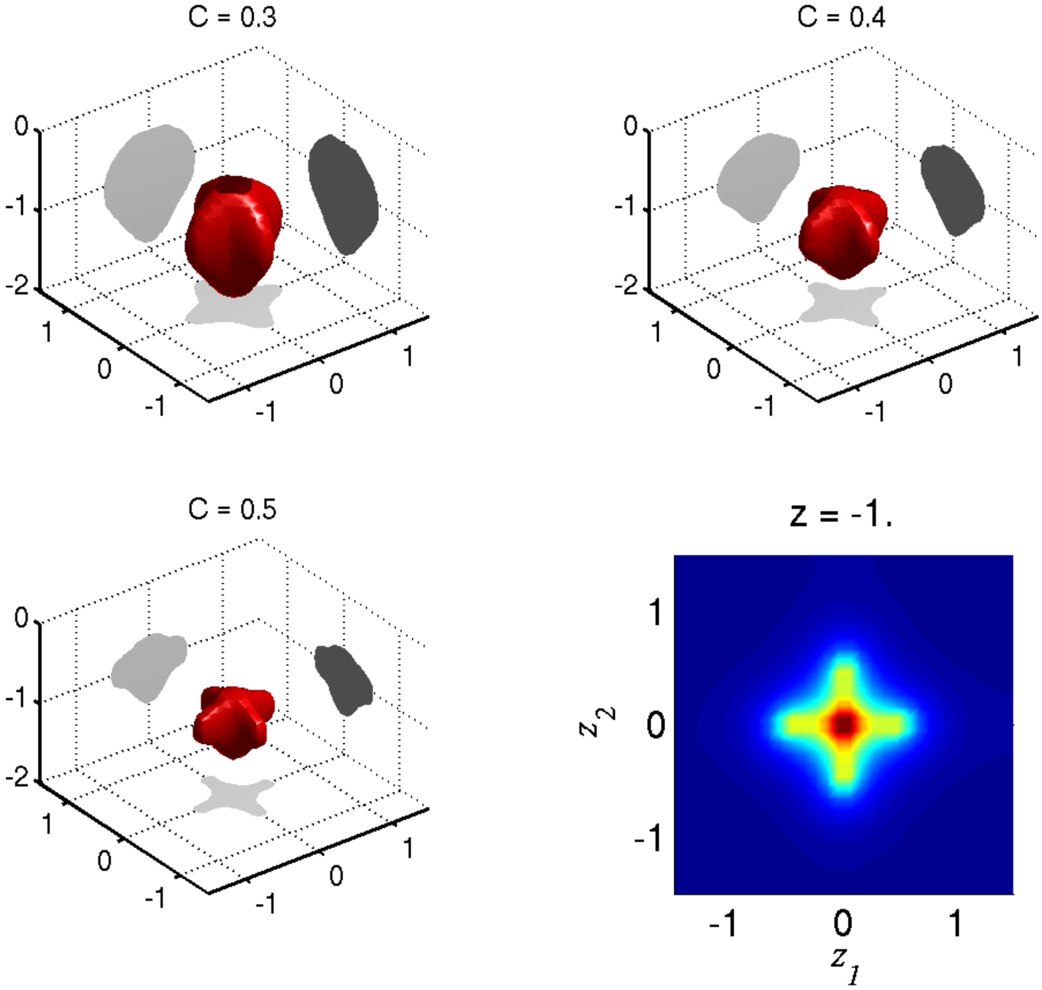
Reconstruction for $n_b = 2 + i$

Numerical Examples ($\mathbb{R}^3$)



Example of a perfectly conducting cross.
The interface earth-air is at $z = 0$. The reconstructions correspond to $n = 2 + 0.5i$ and 1% random noise.

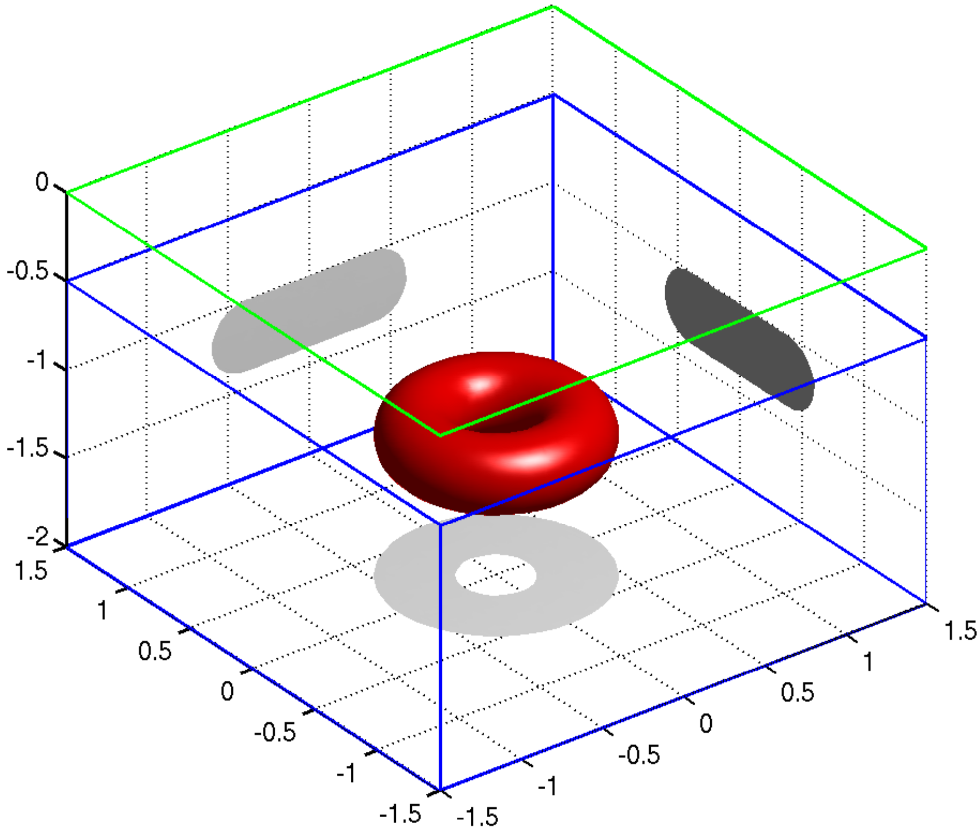
Numerical Examples ($\mathbb{R}^3$)



Reconstruction by using the Reciprocity Gap Functional

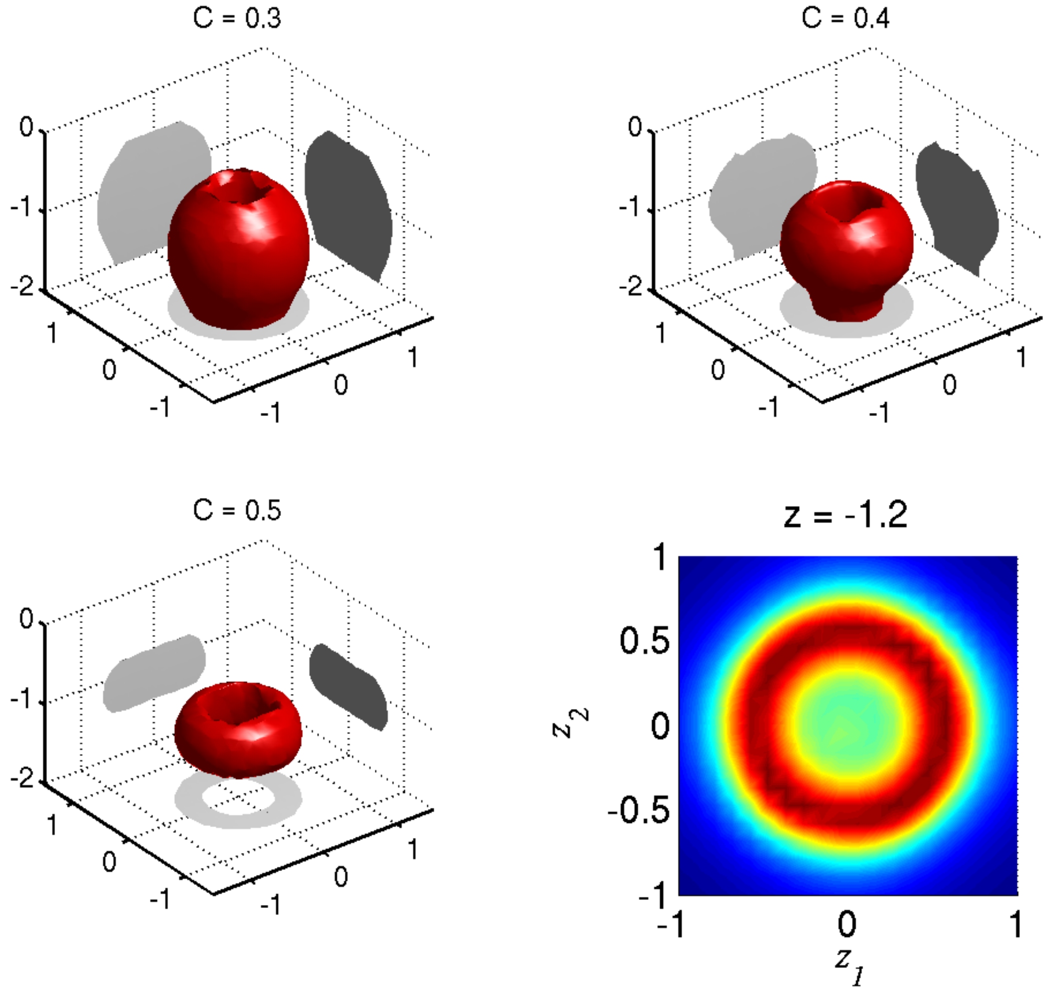
$$k = 2\pi$$

Numerical Examples ($\mathbb{R}^3$)



Example of a perfectly conducting torus.
The interface earth-air is at $z = 0$. The reconstructions correspond to $n = 2 + 0.5i$ and 1% random noise.

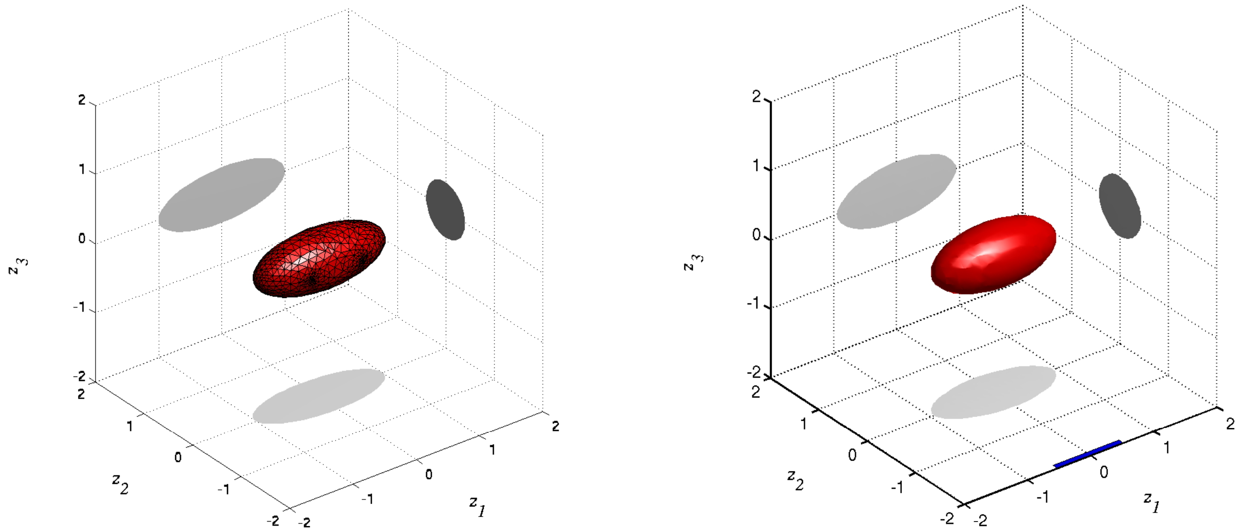
Numerical Examples ($\mathbb{R}^3$)



Reconstruction by using the Reciprocity Gap Functional

$k = 2\pi$

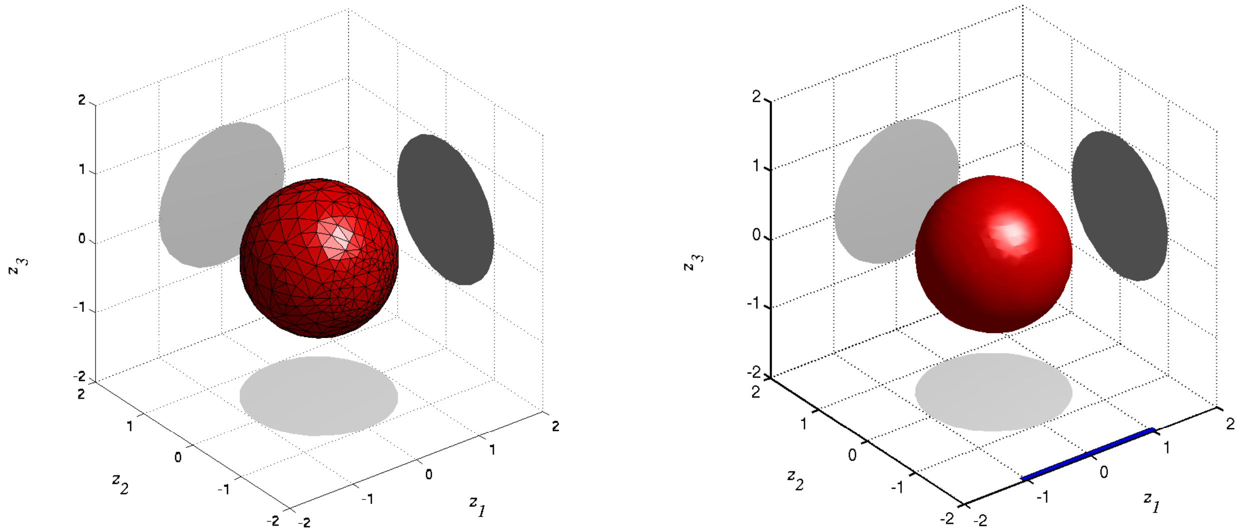
Examples of Reconstructions



Reconstruction of a fully coated ellipsoid with $\eta = 1$ and $k = 6$.

Conducting boundary condition: reconstruction of η			
Exact	Exact ∂D	LSM	LSM/bound
0.0	-0.005	-0.01	-0.004
0.1	0.09	0.16	0.07
1	0.96	0.79	0.58
2	1.15	0.94	0.66

Examples of Reconstructions



Reconstruction of a partially coated sphere. The coated portion Γ_2 is the hemisphere $x_2 > 0$. Here $\eta = 1$ and $k = 3$.

Conducting boundary condition: reconstruction of η			
Exact	Exact Γ_2	LSM	LSM/bound
0.1	0.045	0.037	0.027
1	0.94	0.52	0.43
2	2.00	0.81	0.65

References

1. D. Colton and H. Haddar, An application of the reciprocity gap functional to inverse scattering theory, *Inverse Problems* **21** (2005), 383-398.

2. D. Colton and P. Monk, Target identification of coated objects, *IEEE Transactions on Antennas and Propagation*, to appear.

3. F. Cakoni, M' B Fares and H. Haddar, Analysis of two linear sampling methods applied to electromagnetic imaging of buried objects, to appear.