Some comments on the Born approximation

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Born Scattering Theory

$$\delta G(\mathbf{r},\mathbf{s}) \approx \omega^2 \int G(\mathbf{x},\mathbf{r})\,\delta V(\mathbf{x})\,G(\mathbf{x},\mathbf{s})\,\mathrm{d}\mathbf{x}$$

(just symbolic)

- provides Fréchet differential kernel $\mathrm{d}G(\mathbf{r},\mathbf{s}) = \omega^2 \int G(\mathbf{x},\mathbf{r})\,\mathrm{d}V(\mathbf{x})\,G(\mathbf{x},\mathbf{s})\,\mathrm{d}\mathbf{x}$
  – Green function can be FD
  – basis for most inverse methods (Tarantola, Pratt, Burridge *et al.*)
- allows modelling of scattered waves
  – Green function normally approximated by ray theory

Born Scattering Theory

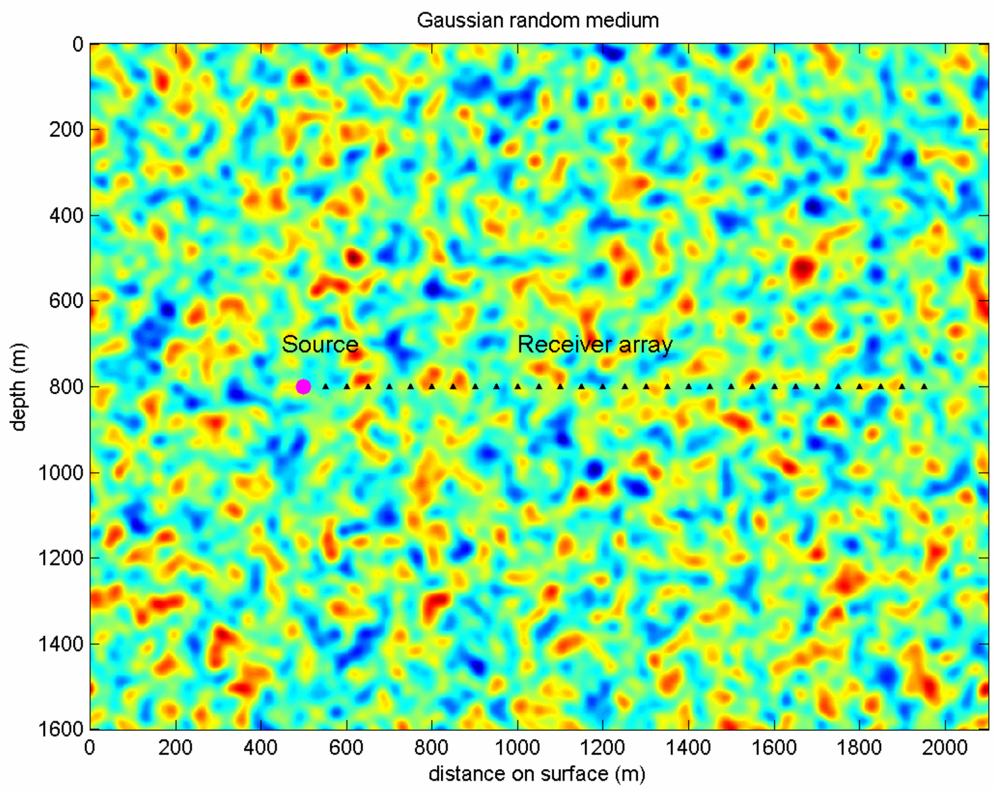
$$\delta G(\mathbf{r},\mathbf{s}) \approx \omega^2 \int A(\mathbf{x},\mathbf{r})\,\delta V(\mathbf{x})\,A(\mathbf{x},\mathbf{s})\,\mathrm{e}^{\mathrm{i}\omega(T(\mathbf{x},\mathbf{r})+T(\mathbf{x},\mathbf{s}))}\mathrm{d}\mathbf{x}$$

- cost in frequency domain $< N_x N_y N_z N_\omega$

$$\delta G(\mathbf{r},\mathbf{s}) \approx -\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int A(\mathbf{x},\mathbf{r})\,\delta V(\mathbf{x})\,A(\mathbf{x},\mathbf{s})\,\delta(t - T(\mathbf{x},\mathbf{r}) - T(\mathbf{x},\mathbf{s}))\,\mathrm{d}\mathbf{x}$$

- cost in time domain $< N_x N_y N_z$
- so in 3D, time domain modelling is attractive

Model and Acquisition Geometry



(courtesy of E. Hopkins and H. Keers, SCR)

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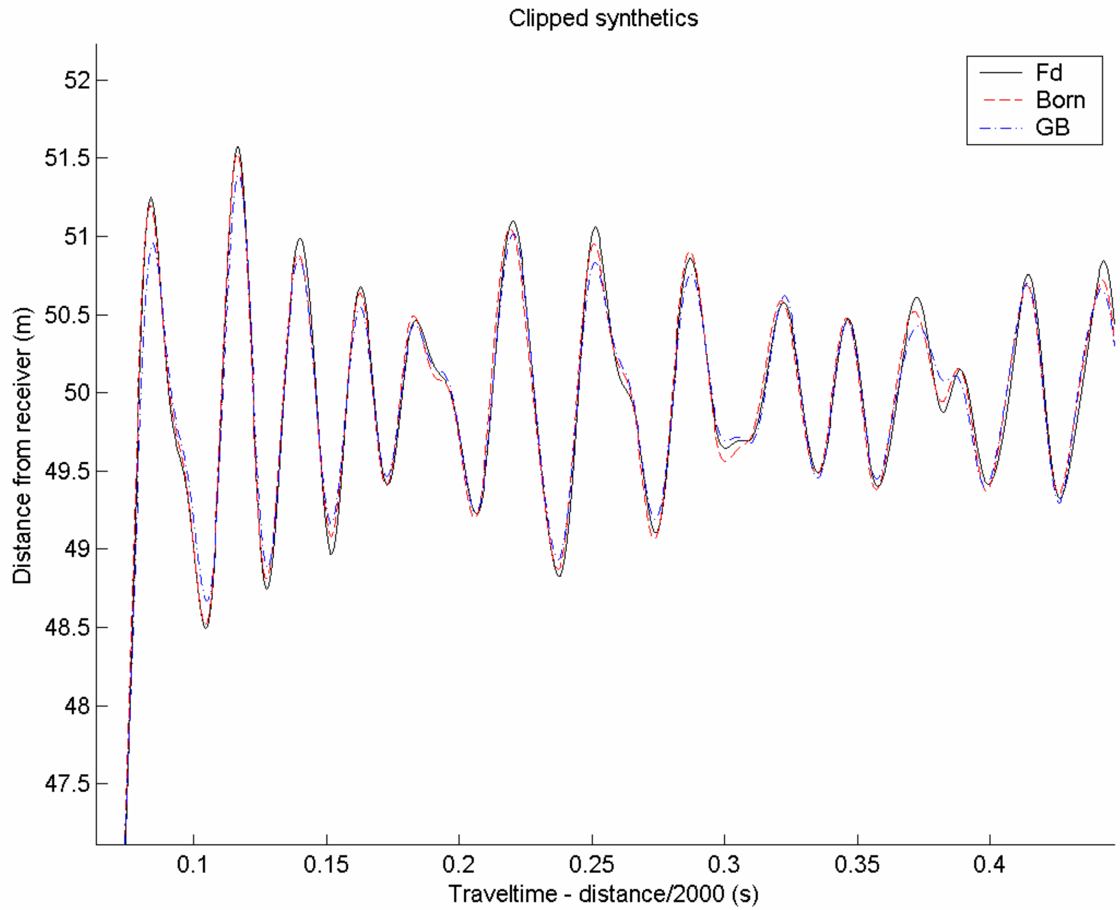
Traces at x = 50m, 2.5%



(courtesy of E. Hopkins and H. Keers, SCR)

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Outline

- standard (perturbation) acoustic Born scattering
  - scattering by model perturbations
  - travel-time perturbations/Fréchet differential kernel
- generalized (error) acoustic Born scattering
  - approximate Green function, e.g. ray theory
  - scattering by errors in wave equation
- comparison with 1D inverse methods
  - inverse method for parameter gradients

Acoustic Born Scattering

- equation of motion

$$-\nabla \underline{P} = -\omega^2 \rho \, \underline{\mathbf{u}} - \mathbf{I} \, \delta(\mathbf{x} - \mathbf{s})$$

press grad　　density x acc　　force

- constitutive equation

$$\nabla . \underline{\mathbf{u}} = -k \underline{P}$$

dilatation　　comp x press

- model perturbation *B* in model *A*

$$\rho = \rho^A + \rho^B$$

$$k = k^A + k^B$$

Acoustic Born Scattering

- solution is known in part *A* of model; part *B* is perturbation
- perturbed equation of motion

$$-\nabla \underline{P}^{A} = -\omega^{2}\rho\,\underline{\mathbf{u}}^{A} - \mathbf{I}\,\delta(\mathbf{x}-\mathbf{s}) + \omega^{2}\rho^{B}\underline{\mathbf{u}}^{A}$$

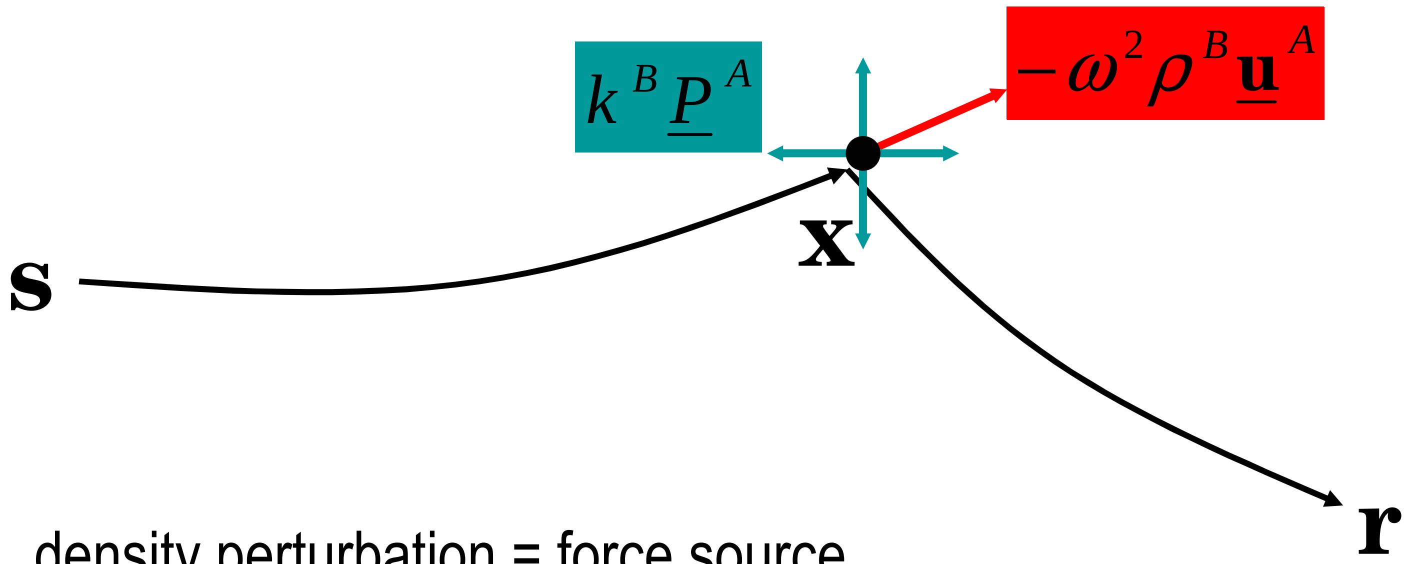
"force source"

- perturbed constitutive equation

$$\nabla.\underline{\mathbf{u}}^{A} = -k\underline{P}^{A} + k^{B}\underline{P}^{A}$$

"press. source"

Point Scatterer



density perturbation = force source

compressibility perturbation = pressure source

Acoustic Born Scattering Integral

- general source using Green (point source) function

$$\mathbf{u}(\omega,\mathbf{r},\mathbf{s}) = \int_V \begin{pmatrix} \underline{\mathbf{u}}^{\mathrm{T}}(\omega,\mathbf{x},\mathbf{r})\,\mathbf{f}_{\mathrm{S}}(\omega,\mathbf{x}) \\ +\,\underline{P}^{\mathrm{T}}(\omega,\mathbf{x},\mathbf{r})\,k_{\mathrm{S}}\,P_{\mathrm{S}}(\omega,\mathbf{x}) \end{pmatrix} \mathrm{d}V$$

volume integral of sources

- scattering source

$$\underline{\mathbf{u}}^{B}(\omega,\mathbf{r},\mathbf{s}) = \int_V \begin{pmatrix} \underline{\mathbf{u}}^{\mathrm{T}}(\omega,\mathbf{x},\mathbf{r})\,\omega^2\rho^{B}(\mathbf{x})\underline{\mathbf{u}}^{A}(\omega,\mathbf{x},\mathbf{s}) \\ +\,\underline{P}^{\mathrm{T}}(\omega,\mathbf{x},\mathbf{r})\,k^{B}(\mathbf{x})\underline{P}^{A}(\omega,\mathbf{x},\mathbf{s}) \end{pmatrix} \mathrm{d}V$$

volume integral of scattering sources

- note $\underline{\mathbf{u}} = \underline{\mathbf{u}}^{A} + \underline{\mathbf{u}}^{B}$ in integral

Acoustic Born Scattering Approximation

- approximate full Green function by Green function in reference medium

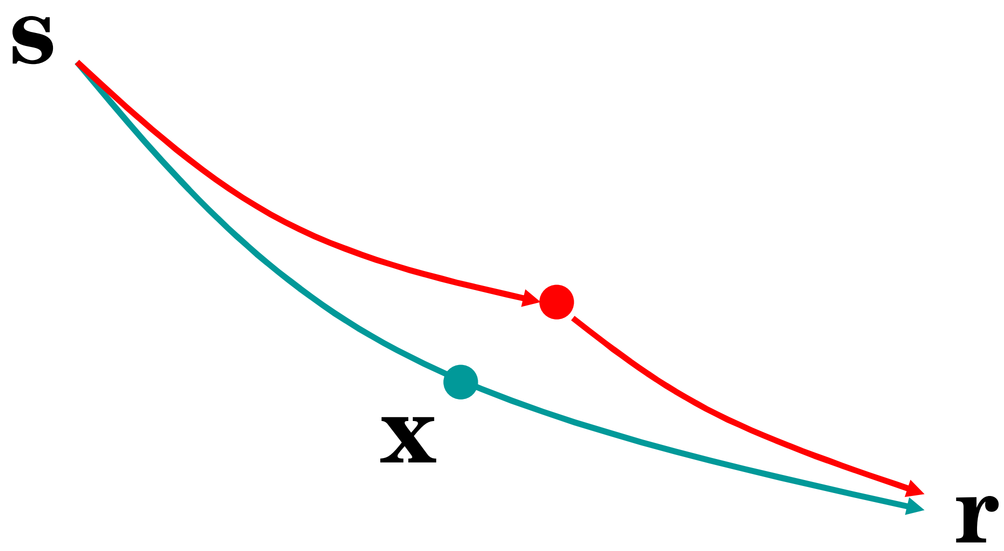
$$\underline{\mathbf{u}}^{B}(\omega,\mathbf{r},\mathbf{s}) \approx \int_{V} \begin{pmatrix} \underline{\mathbf{u}}^{A^{\mathrm{T}}}(\omega,\mathbf{x},\mathbf{r})\,\omega^{2}\rho^{B}(\mathbf{x})\underline{\mathbf{u}}^{A}(\omega,\mathbf{x},\mathbf{s}) \\ +\underline{P}^{A^{\mathrm{T}}}(\omega,\mathbf{x},\mathbf{r})\,k^{B}(\mathbf{x})\underline{P}^{A}(\omega,\mathbf{x},\mathbf{s}) \end{pmatrix} \mathrm{d}V$$

volume integral of scattering sources
with Green function in reference medium

Pros and Cons

- excellent if perturbations are small and isolated
- but
  - assumes Green function known exactly
  - assumes perturbation has little effect on Green function
- what if Green function is approximate?
- in general, how do we choose choose perturbation?
- what if perturbations are extensive -
  how are ray results perturbed?

The Ray Theory Approximation in the Born Volume



$$G(\omega, \mathbf{x}, \mathbf{s}) = A(\mathbf{x}, \mathbf{s})\, e^{i\omega T(\mathbf{x}, \mathbf{s})}$$

Ray Theory - an Approximate Green function

transmission dyadic

$$\begin{pmatrix} -i\omega\, \underline{\mathbf{u}}^{A} \\ -\underline{P}^{A} \end{pmatrix} = -\frac{\mathrm{i}\omega}{2\pi} e^{i\omega T}\, \mathscr{T}(\mathbf{x},\mathbf{s}) \begin{pmatrix} \mathbf{g} \\ -(Z/2)^{1/2} \end{pmatrix}(\mathbf{x})\ \mathbf{g}^{\mathrm{T}}(\mathbf{s})$$

$$Z = \rho\alpha$$ impedance

$$\mathbf{g} = (2Z)^{-1/2}\,\hat{\mathbf{g}}$$ energy-normalized polarization

$$\mathscr{T}(\mathbf{x},\mathbf{s}) = \mathscr{S}^{-1/2}(\mathbf{x},\mathbf{s})\, e^{-\mathrm{i}\pi\sigma(\mathbf{x},\mathbf{s})/2}$$ transmission function

$$\mathscr{S}(\mathbf{x},\mathbf{s}) = \left|\frac{\partial\mathbf{x}}{\partial q_1}\times\frac{\partial\mathbf{x}}{\partial q_2}\right| \Big/ \left|\frac{\partial\mathbf{p}_s}{\partial q_1}\times\frac{\partial\mathbf{p}_s}{\partial q_2}\right| = [c(\mathbf{s})\mathscr{R}(\mathbf{x},\mathbf{s})]^2$$ spreading function

Acoustic Born Scattering Approximation
frequency domain

$$\underline{\mathbf{u}}^{B}(\omega,\mathbf{r},\mathbf{s}) \approx \frac{\omega^{2}}{4\pi^{2}} \int_{V} \mathbf{K}^{B}(\mathbf{x},\mathbf{r},\mathbf{s})\, e^{\mathrm{i}\omega T(\mathbf{x},\mathbf{r},\mathbf{s})} \mathrm{d}V$$

$$\mathbf{K}^{B}(\mathbf{x},\mathbf{r},\mathbf{s}) = -\mathbf{g}(\mathbf{r})\, \mathscr{T}(\mathbf{x},\mathbf{r})\, \Gamma^{B}(\mathbf{x},\mathbf{r},\mathbf{s})\, \mathscr{T}(\mathbf{x},\mathbf{s})\, \mathbf{g}^{\mathrm{T}}(\mathbf{s})$$

$$\Gamma^{B}(\mathbf{x},\mathbf{r},\mathbf{s}) = \mathbf{g}^{\mathrm{T}}(\mathbf{x},\mathbf{r})\, \rho^{B}(\mathbf{x})\, \mathbf{g}(\mathbf{x},\mathbf{s}) - \frac{1}{2} Z(\mathbf{x})\, k^{B}(\mathbf{x})$$

scattering scalar amplitude (independent of frequency)

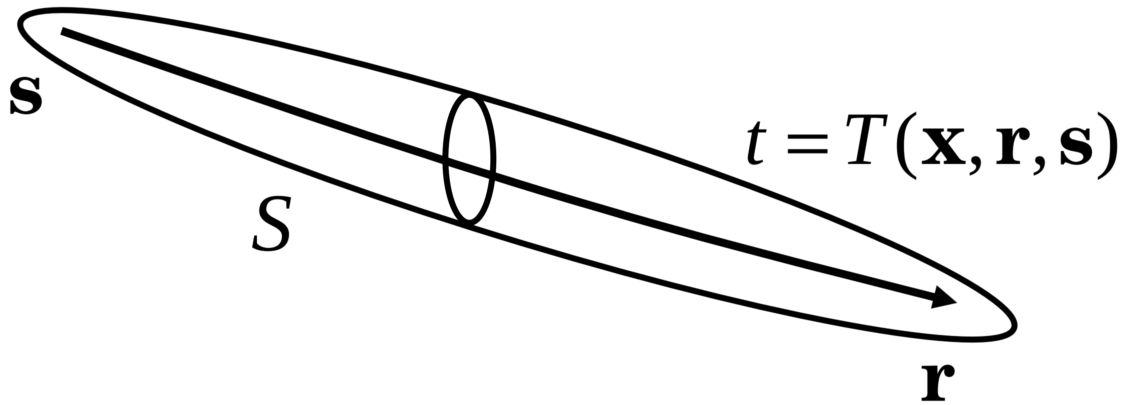
$$T(\mathbf{x},\mathbf{r},\mathbf{s}) = T(\mathbf{x},\mathbf{r}) + T(\mathbf{x},\mathbf{s})$$

total scattered travel time

Acoustic Born Scattering Approximation
time domain

$$\underline{\mathbf{u}}^{B}(t,\mathbf{r},\mathbf{s}) \approx -\frac{1}{4\pi^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_V \mathbf{K}^{B}(\mathbf{x},\mathbf{r},\mathbf{s})\,\delta(t-T(\mathbf{x},\mathbf{r},\mathbf{s}))\mathrm{d}V$$

$$= -\frac{1}{4\pi^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_S \frac{\mathbf{K}^{B}(\mathbf{x},\mathbf{r},\mathbf{s})}{\left|\nabla T(\mathbf{x},\mathbf{r},\mathbf{s})\right|}\mathrm{d}S$$



Born Kernel for Travel Time

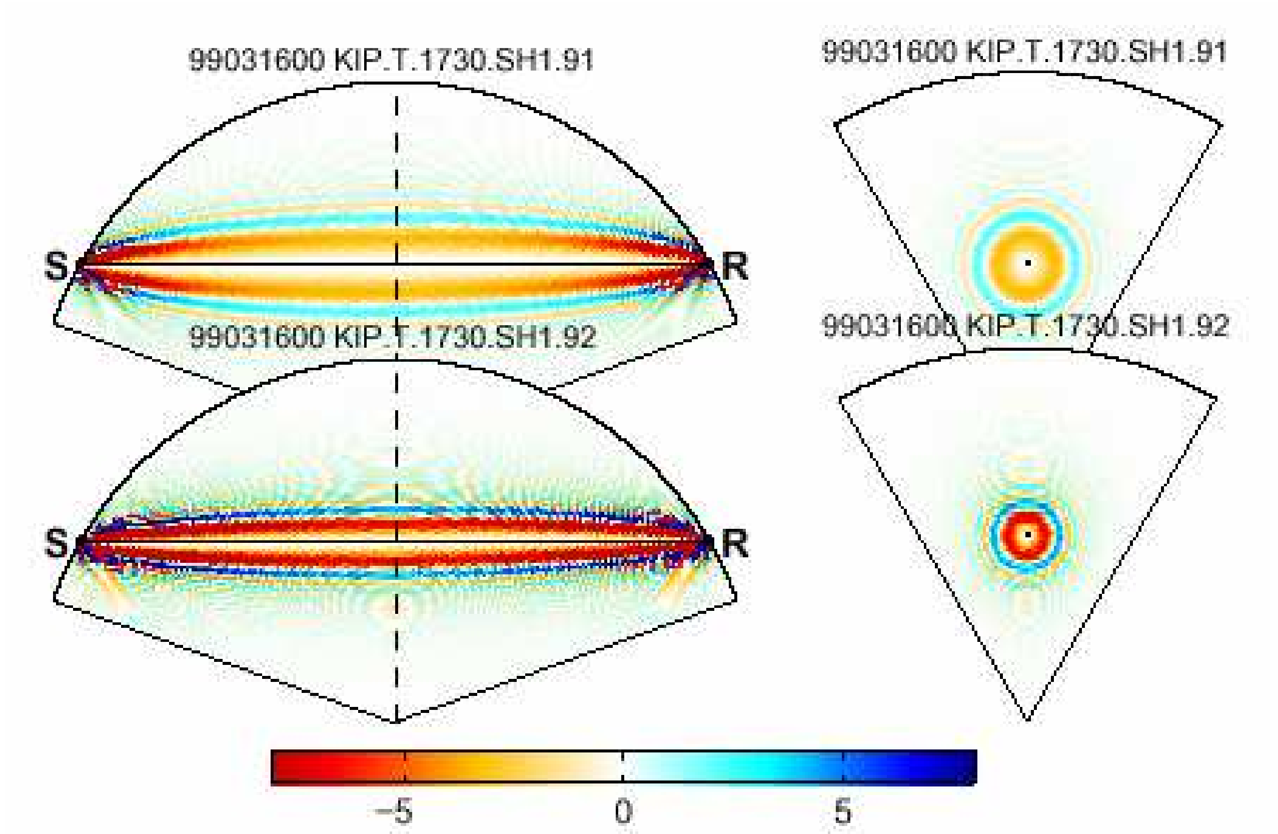
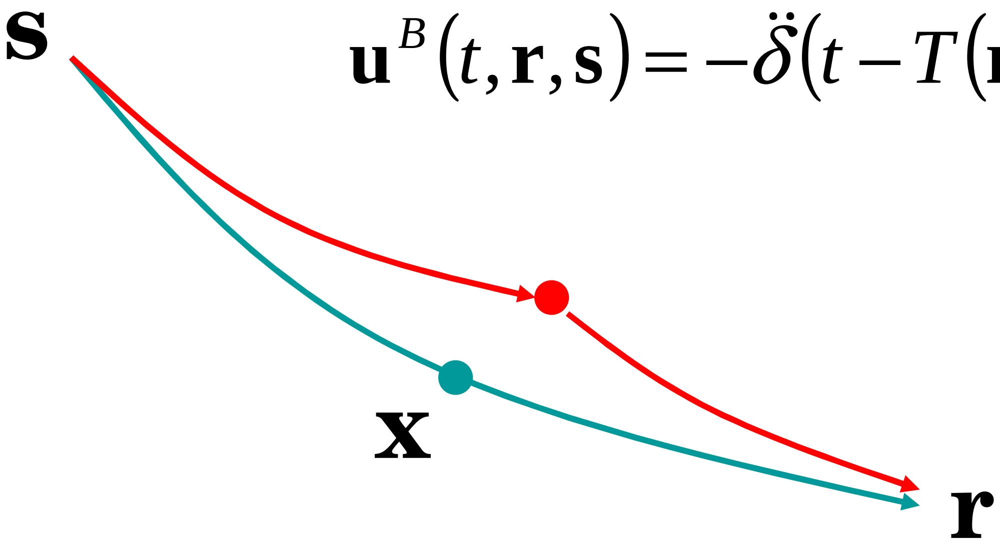


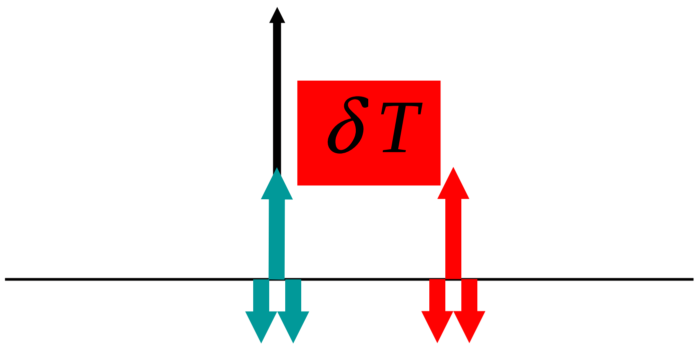
Fig.2

L. Zhao, T.H. Jordan and C.H. Chapman, 2000.
Three-dimensional Frechet differential kernels for seismic delay times,
Geophys. J. Intl., **141**, 558-576.

Point Perturbation On and Off Ray

$$\mathbf{u}^B(t,\mathbf{r},\mathbf{s}) = -\ddot{\delta}\left(t - T(\mathbf{r},\mathbf{s}) - \delta T\right)\mathbf{K}^B(\mathbf{x},\mathbf{r},\mathbf{s})\,\Delta V$$





- Born scattering is second derivative of incident pulse
- on-ray scattering, pulse only distorted, not delayed
- traveltime perturbation is due to perturbations *off* the ray (within Fresnel zone), integrating laterally to give first derivative of incident pulse

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Born Scattering Theory for Modelling

$$\delta G(\mathbf{r},\mathbf{s}) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int A(\mathbf{x},\mathbf{r})\ \delta V(\mathbf{x})\ A(\mathbf{x},\mathbf{s})\ \delta(t - T(\mathbf{x},\mathbf{r}) - T(\mathbf{x},\mathbf{s}))\ \mathrm{d}\mathbf{x}$$

- efficient in time domain
- efficient for small isolated scatterers
- intuitively not physical for extended scatterers
  - scattering should be from gradients not perturbations
- only models travel time correction as $\mathrm{e}^{\mathrm{i}\,\omega\,\delta \mathrm{T}} \approx 1 + \mathrm{i}\,\omega\,\delta\mathrm{T}$ for extended scatterers

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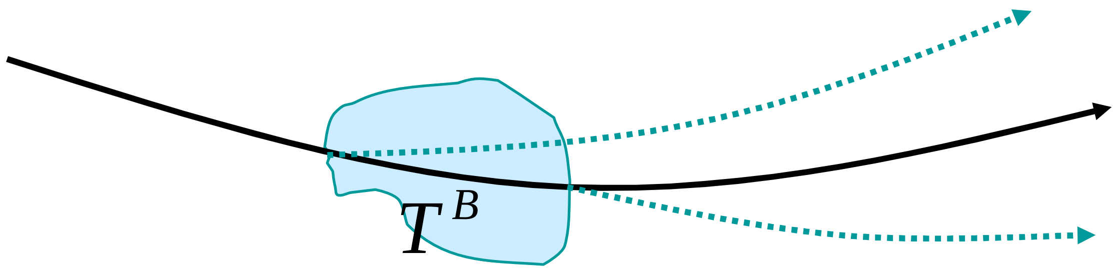
Born Scattering Dichotomy

- for an extended perturbation, the Born approximation must correct travel times, plus model scattering from edges of perturbation
- it gives a perturbation $\underline{\mathbf{u}}^{B}(\omega,\mathbf{r},\mathbf{s}) \approx -\mathrm{i}\omega\, \underline{\mathbf{u}}^{A}(\omega,\mathbf{r},\mathbf{s})\; T^{B}$ which at high frequencies is a *very poor* approximation
- if the perturbation is halved, $k^{A} + k^{B}/2 \Rightarrow k^{A},\; k^{B}/2 \Rightarrow k^{B}$ the travel-time error is reduced, but the scattered signal from the edges is also reduced!



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Generalized Born Scattering

- recall wave equation with perturbation

$$-\nabla \underline{P}^{A} = -\omega^{2}\rho\,\underline{\mathbf{u}}^{A} - \mathbf{I}\delta(\mathbf{x}-\mathbf{s}) + \omega^{2}\rho^{B}\underline{\mathbf{u}}^{A}$$

- suppose $\underline{P}^{A}$ and $\underline{\mathbf{u}}^{A}$ are *not* exact solutions in model A
- error in equation of motion

$$\underline{\mathbf{F}}^{A} = \omega^{2}\rho^{A}\underline{\mathbf{u}}^{A} - \nabla \underline{P}^{A} + \mathbf{I}\delta(\mathbf{x}-\mathbf{s}) \neq \underline{0}$$

- similarly in constitutive equation error is

$$\underline{\Theta}^{A} = \nabla.\underline{\mathbf{u}}^{A} + k^{A}\underline{P}^{A} \neq \underline{0}$$

- note these are definitions of errors not differential equations

Generalized Born Scattering

- modifed equaton of motion and constitutive equation

$$-\nabla \underline{P}^{A} = -\omega^{2} \rho \, \underline{\mathbf{u}}^{A} - \mathbf{I}\delta(\mathbf{x}-\mathbf{s}) + \underline{\mathscr{E}}^{N}$$

error + perturbation

$$\nabla . \underline{\mathbf{u}}^{A} = -k \underline{P}^{A} + \underline{\mathscr{E}}^{H}$$

similarly

- where

$$\underline{\mathscr{E}}^{N} = \underline{\mathbf{F}}^{A} + \omega^{2} \rho^{B} \, \underline{\mathbf{u}}^{A}$$

$$\underline{\mathscr{E}}^{H} = \underline{\Theta}^{A} + k^{B} \underline{P}^{A}$$

errors perturbations

Generalized Born Scattering

- standard Born scattering (perturbations are sources)

$$\underline{\mathbf{u}}^{B}(\omega,\mathbf{r},\mathbf{s})=\int_{V}\begin{pmatrix}\underline{\mathbf{u}}^{\mathrm{T}}(\omega,\mathbf{x},\mathbf{r})\,\omega^{2}\rho^{B}(\mathbf{x})\underline{\mathbf{u}}^{A}(\omega,\mathbf{x},\mathbf{s})\\+\underline{P}^{\mathrm{T}}(\omega,\mathbf{x},\mathbf{r})\,k^{B}(\mathbf{x})\underline{P}^{A}(\omega,\mathbf{x},\mathbf{s})\end{pmatrix}\mathrm{d}V$$

- generalized Born scattering (errors+perturbations are sources)

$$\underline{\mathbf{u}}^{B}(\omega,\mathbf{r},\mathbf{s})=\int_{V}\begin{pmatrix}\underline{\mathbf{u}}^{\mathrm{T}}(\omega,\mathbf{x},\mathbf{r})\,\underline{\mathscr{E}}^{N}(\omega,\mathbf{x},\mathbf{s})\\+\underline{P}^{\mathrm{T}}(\omega,\mathbf{x},\mathbf{r})\,\underline{\mathscr{E}}^{H}(\omega,\mathbf{x},\mathbf{s})\end{pmatrix}\mathrm{d}V$$

Generalized Born Approximation

- ray approximation

$$\begin{pmatrix} -i\omega\,\underline{\mathbf{u}}^{A} \\ -\underline{P}^{A} \end{pmatrix} = -\frac{i\omega}{2\pi} e^{i\omega T} \begin{pmatrix} \underline{\mathbf{v}}^{(0)} \\ -\underline{P}^{(0)} \end{pmatrix} = -\frac{i\omega}{2\pi} e^{i\omega T} \mathscr{T}(\mathbf{x},\mathbf{s}) \begin{pmatrix} \mathbf{g} \\ -(Z/2)^{1/2} \end{pmatrix}(\mathbf{x})\ \mathbf{g}^{\mathrm{T}}(\mathbf{s})$$

- error terms

$$\underline{\mathbf{F}}^{A} = \frac{i\omega}{2\pi} e^{i\omega T}\ \nabla \underline{P}^{(0)}$$

$$\underline{\Theta}^{A} = \frac{1}{2\pi} e^{i\omega T}\ \nabla.\underline{\mathbf{v}}^{(0)}$$

Generalized Born Approximation

time domain

$$\underline{\mathbf{u}}^{B}(t,\mathbf{r},\mathbf{s}) \approx -\frac{1}{4\pi^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_S \frac{\mathbf{K}^{B}(\mathbf{x},\mathbf{r},\mathbf{s})}{\left|\nabla T(\mathbf{x},\mathbf{r},\mathbf{s})\right|}\,\mathrm{d}S$$

perturbation scattering

$$+\frac{1}{4\pi^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_S \frac{\mathbf{K}^{E}(\mathbf{x},\mathbf{r},\mathbf{s})}{\left|\nabla T(\mathbf{x},\mathbf{r},\mathbf{s})\right|}\,\mathrm{d}S$$

error scattering

Generalized Born Approximation

- acoustic scattering kernel

$$
\begin{aligned}
\mathbf{K}^{E}(\mathbf{x},\mathbf{r},\mathbf{s}) &= -\underline{\mathbf{v}}_{\mathrm{R}}^{(0)\mathrm{T}}\nabla \underline{P}_{\mathrm{S}}^{(0)} + \underline{P}_{\mathrm{R}}^{(0)\mathrm{T}}\nabla .\underline{\mathbf{v}}_{\mathrm{S}}^{(0)} \\
&= -\frac{1}{2}\left[\underline{\mathbf{v}}_{\mathrm{R}}^{(0)\mathrm{T}}\nabla \underline{P}_{\mathrm{S}}^{(0)} - \underline{P}_{\mathrm{R}}^{(0)\mathrm{T}}\nabla .\underline{\mathbf{v}}_{\mathrm{S}}^{(0)} + \left(\nabla \underline{P}_{\mathrm{R}}^{(0)}\right)^{\mathrm{T}} \underline{\mathbf{v}}_{\mathrm{S}}^{(0)} - \left(\nabla .\underline{\mathbf{v}}_{\mathrm{R}}^{(0)}\right)^{\mathrm{T}} \underline{P}_{\mathrm{S}}^{(0)}\right] \\
&\approx -\mathbf{g}(\mathbf{r})\ \mathscr{T}_{\mathrm{R}}\ \Gamma^{E}\ \mathscr{T}_{\mathrm{S}}\ \mathbf{g}^{\mathrm{T}}(\mathbf{s})
\end{aligned}
$$

(ignoring derivatives of source and receiver directivity)

Generalized Born Approximation

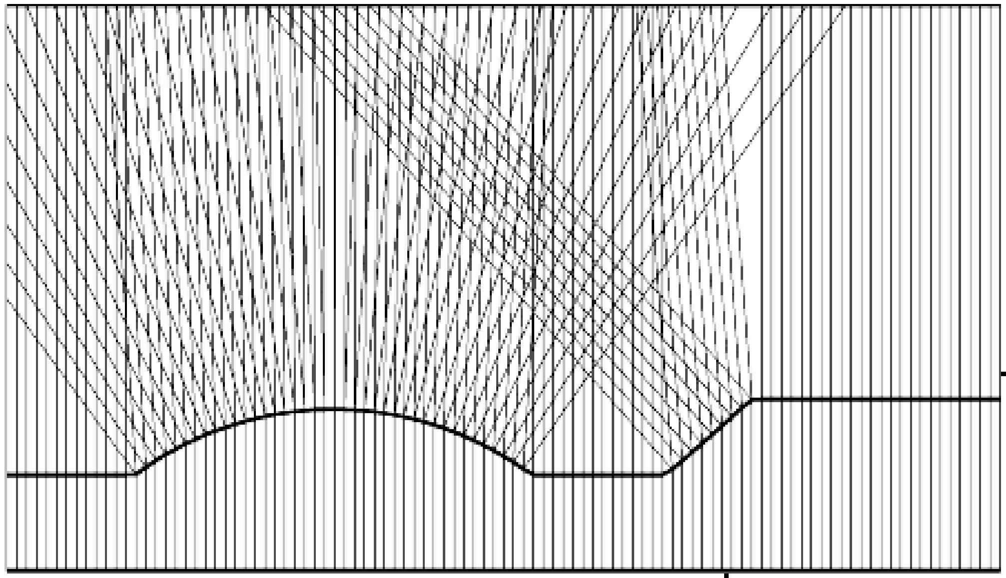
- scalar ray acoustic scattering kernel

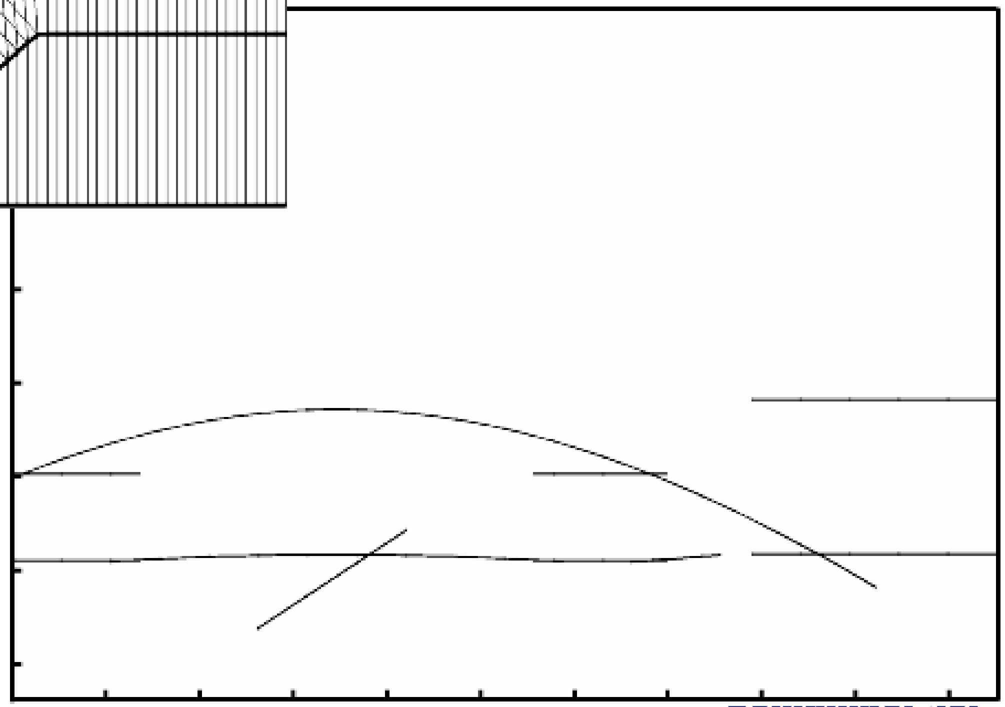
$$\Gamma^{E}(\mathbf{x},\mathbf{r},\mathbf{s}) = \frac{1}{4}\left(\frac{\nabla Z}{Z} - \nabla\right).\left(\hat{\mathbf{g}}_{R} + \hat{\mathbf{g}}_{S}\right) - \frac{1}{4}\left(\nabla \ln \frac{\mathscr{T}_{R}}{\mathscr{T}_{S}}\right).\left(\hat{\mathbf{g}}_{R} - \hat{\mathbf{g}}_{S}\right)$$

$$= \frac{1}{4}\left(\hat{\mathbf{g}}_{R} + \hat{\mathbf{g}}_{S}\right).\nabla \ln\left(Z \mathscr{T}_{R} \mathscr{T}_{S}\right)$$

using energy conservation $\nabla.\left(\mathscr{T}^{2}\mathbf{g}\right) = 0$

‘2D’ Profile in French Model



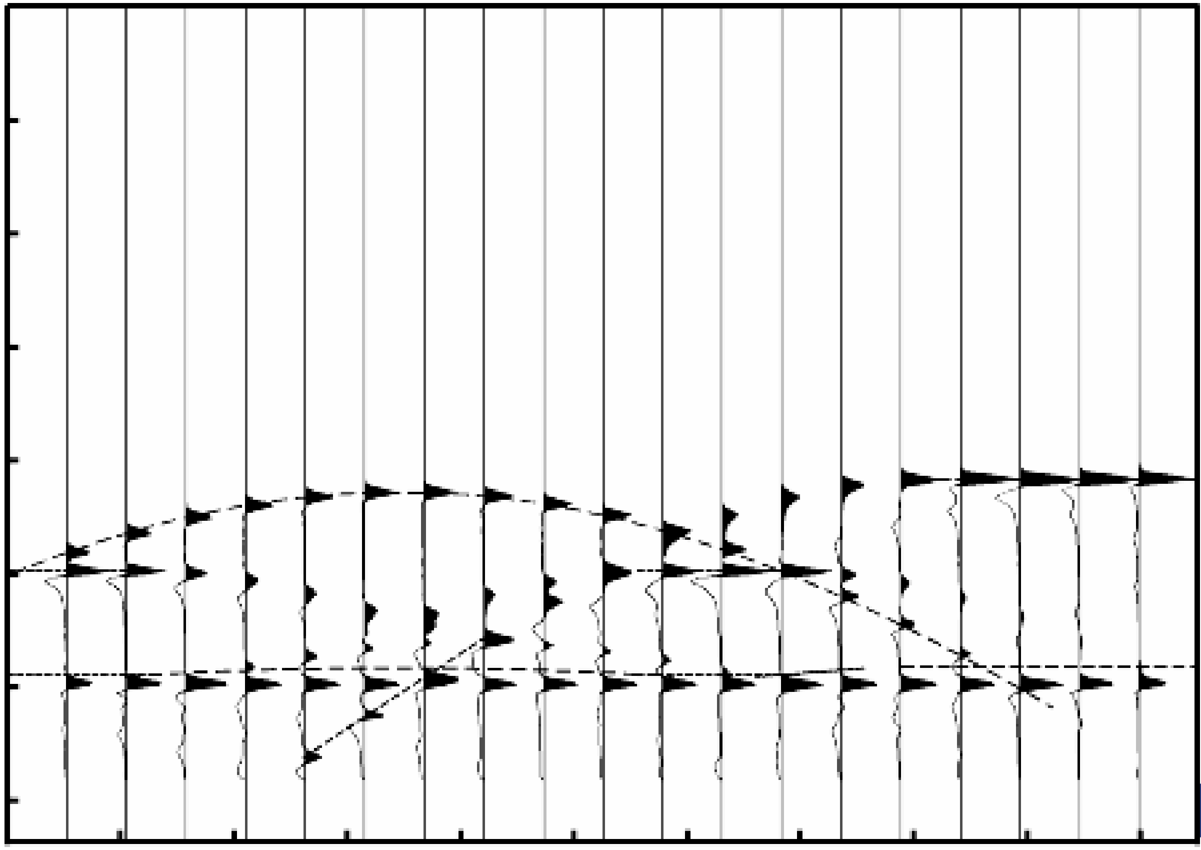


French, 1974

Born Perturbation in French Model

– volume scattering from model perturbation

$$\mathbf{u} = \frac{1}{4\pi^2}\frac{d^2}{dt^2}\int_V \mathrm{Re}\left[\Delta\left(t-\tilde{T}\right)\mathbf{K}\right]dV \qquad \Gamma^B = \rho^B\,\mathbf{g}_{\mathrm{R}}^T\mathbf{g}_{\mathrm{S}} - k^B\,Z/2$$



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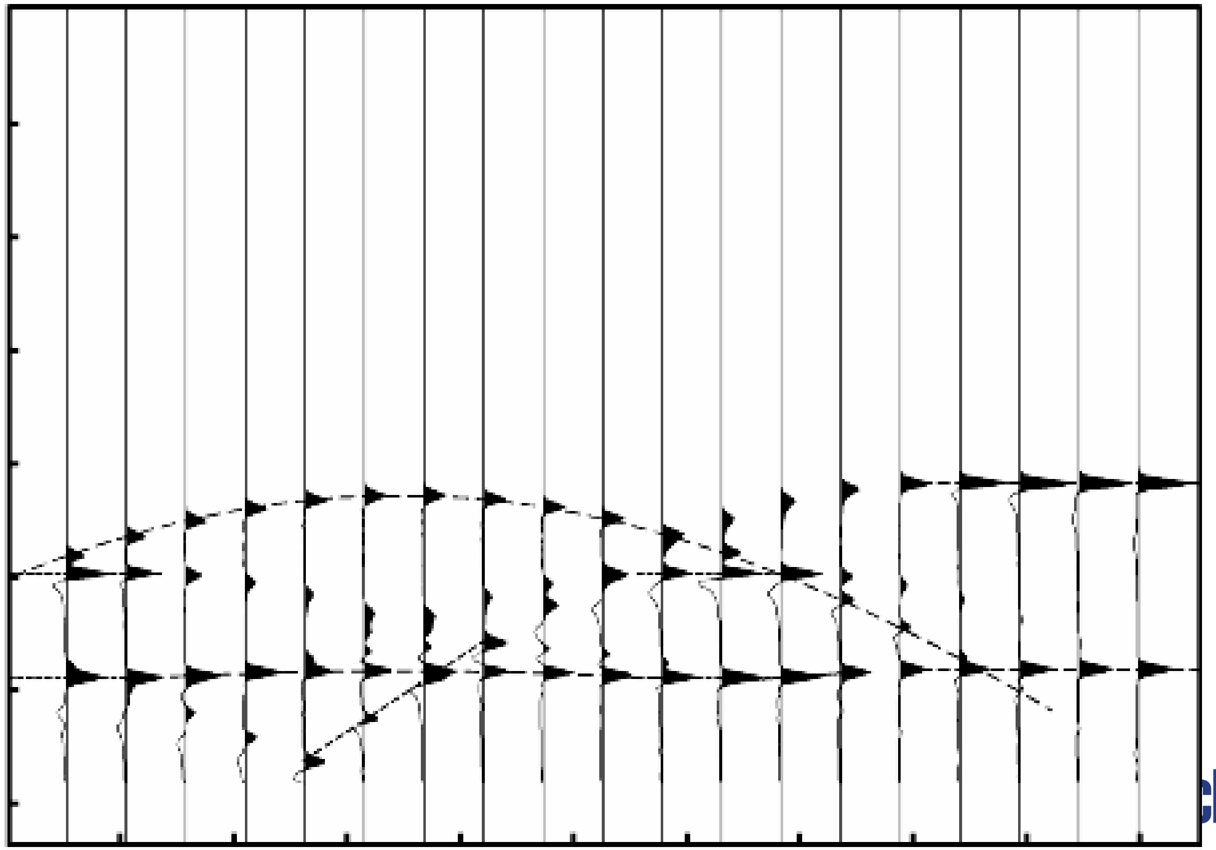
Born Error in French Model

– volume scattering from ray theory errors

$$\mathbf{u} = -\frac{1}{4\pi^2}\frac{d}{dt}\int_V \mathrm{Re}\left[\Delta\left(t-\tilde{T}\right)\mathbf{K}\right]dV \qquad \Gamma^E = \frac{1}{4}\left(\hat{\mathbf{g}}_\mathrm{R} + \hat{\mathbf{g}}_\mathrm{S}\right).\nabla \ln\left(\mathrm{Z}\mathscr{T}_\mathrm{R}\mathscr{T}_\mathrm{S}\right)$$



Elastic Generalized Born

- just as acoustic generalized Born except

$$\Gamma^{B}(\mathbf{x},\mathbf{r},\mathbf{s})\text{and} \qquad \Gamma^{E}(\mathbf{x},\mathbf{r},\mathbf{s})$$

are more complicated

Generalized Born Approximation

- references
  - C.H. Chapman and R.T. Coates, 1994.
    Generalized Born scattering in anisotropic media,
    Wave Motion, **19**, 309-341.
  - R.T.Coates and C.H.Chapman, 1991.
    Generalized Born scattering of elastic waves in 3-D media,
    Geophys.J.Int.,**107**, 231-263.
  - C.H.Chapman, 2004.
    Fundamentals of Seismic Wave Propagation,
    Cambridge University Press.

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cf. Transformed Wave Equation in 1D Model

- to understand generalized Born
- to investigate the inverse problem

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{dz}} = \mathrm{i}\omega \mathbf{A}\,\mathbf{w}$$

1D differential wave equation
in (ω, p, z) domain

$$\mathbf{w} = \begin{pmatrix} \mathrm{v_z} \\ -P \end{pmatrix}$$

variables

$$\mathbf{A}\,\mathbf{W} = \mathbf{W}\,\mathbf{p}$$

eigen-equation (U/D separation)

$$\tau = \int^{z} \mathbf{p}\,\mathrm{d}\varsigma$$

vertical delay time

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Bremmer Coupling Equations

$$\mathbf{w} = \mathbf{W}\exp(\mathrm{i}\omega\tau)\,\mathbf{r}$$ resolve into U/D components

$\mathbf{W}\exp(\mathrm{i}\omega\tau)$ = "rays" $\mathbf{r}$ = "ray amplitudes"

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}z} = \exp(-\mathrm{i}\omega\tau)\,\mathbf{C}\exp(\mathrm{i}\omega\tau)\,\mathbf{r}$$ coupling differential equation

$$\mathbf{C} = -\mathbf{W}^{-1}\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}z}$$ coupling differential coefficients

$$\mathbf{r}^{(n+1)} = \mathbf{r}^{(n)} + \int^{z} \exp(-\mathrm{i}\omega\tau)\,\mathbf{C}\exp(\mathrm{i}\omega\tau)\,\mathbf{r}^{(n)}\,\mathrm{d}\varsigma$$ iterative coupling

"Born error series" $\mathbf{C} \to \Gamma^{E}$

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Differential Coupling Coefficients

$$\mathbf{C} = -\mathbf{W}^{-1} \frac{\mathrm{d}\mathbf{W}}{\mathrm{d}z} = \begin{pmatrix} 0 & \gamma_A \\ \gamma_A & 0 \end{pmatrix}$$

$$\gamma_A = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}z} \ln\left(\frac{\rho}{q_\alpha} \right)$$

- 1D Bremmer iterative solution
  - H. Bremmer, 1949
    Terrestrial Radio Waves
    Elsevier Publishing Company, p. 159
  - J.G.J. Scholte, 1962
    Oblique propagation of waves in inhomogeneous media
    Geophys. J.R. astr. Soc., **7**, 244-261

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1D Inverse Problem

- classic solutions
  - Gel'fand & Levitan (1951), Marchenko (1955), Gopinath-Sondhi (1971), Blagoveschenskiy (1978), …
    (R. Burridge, 1980. The Gelfand-Levitan, the Marchenko, and the Gopinath-Sondhi integral equation of inverse scattering theory, regarded in the context of inverse-reponse problems, *Wave Motion*, **2**, 305-323)

- K. Bube and R. Burridge, 1982. The one-dimensional inverse problem of reflection seismology, *SIAM Rev*., **25**, 497-559.
  - downward propagate wave variables $\mathbf{w}$
  - causal point gives reflected wave
  - ratio of components of wave variables gives $\mathrm{v}_z / P = q_\alpha / \rho$
    material properties

1D Inverse Problem – *Bailey's method*

- R.C. Bailey, 1970. *Some inverse problems in geophysics*, Ph.D. thesis, Cambridge University.
  - N.D. Bregman, C.H. Chapman and R.C. Bailey, 1985. A non-iterative procedure for inverting plane-wave reflection data at several angles of incidence using the Ricatti Equation,
    Geophys. Prosp., **33**, 185-200.
    - downward propagate reflectivity $R = r_1 / r_2$
    - causal point gives differential reflection coefficient γ_A
- ray continuation
    - downward propagate ray amplitudes $\mathbf{r}$
    - causal point gives differential reflection coefficient γ_A

Ricatti Reflectivity Equation

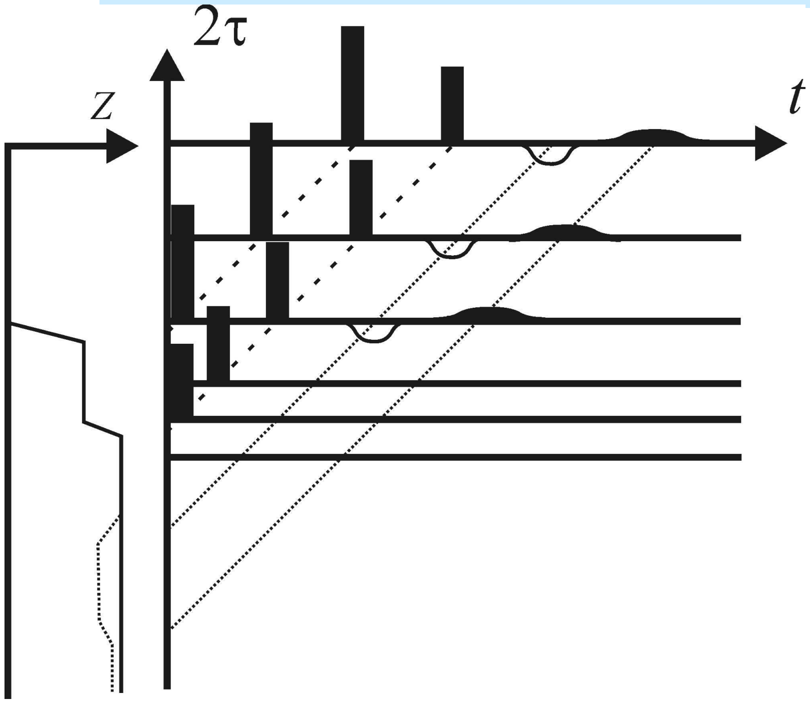
$$R(\omega, p, \tau) = \frac{r_1(\omega, p, \tau)}{r_2(\omega, p, \tau)} \equiv$$ reflectivity (up/down)

$r_1(t, p, 0) \equiv$ data

$r_2(t, p, 0) \equiv$ source

$$\frac{1}{2}\frac{\mathrm{d}R}{\mathrm{d}\tau} = \mathrm{i}\,\omega R + \gamma_A (1 - R^2)$$ Ricatti equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau} R(t, p, \tau) = -\frac{\mathrm{d}}{\mathrm{d}t} R(t, p, \tau) + \gamma_A(p, \tau)\big(\delta(t) - R(t, p, \tau) * R(t, p, \tau)\big)$$



$$\gamma_A = -\frac{1}{2}\frac{\partial}{\partial \tau} \ln\left(\frac{\rho}{q_\alpha}\right) \equiv$$ diff. reflection coeff.

$$R(0-, p, \tau) = 0$$ causality

$$R^{(1)}(t, p, \tau) = \gamma_A(p, 2\tau - t) H(t)$$ 1st iteration

$$r_1(t, p, 0) = r_2(t, p, 0) * \gamma_A(p, -t) H(t)$$ convolution model

$$\gamma_A(p, 2\tau) = R(0+, p, \tau)$$ inverse solution

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Differential Coupling Coefficients

$$\mathbf{C} = -\mathbf{W}^{-1}\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}z} = \begin{pmatrix} 0 & \gamma_A \\ \gamma_A & 0 \end{pmatrix}$$

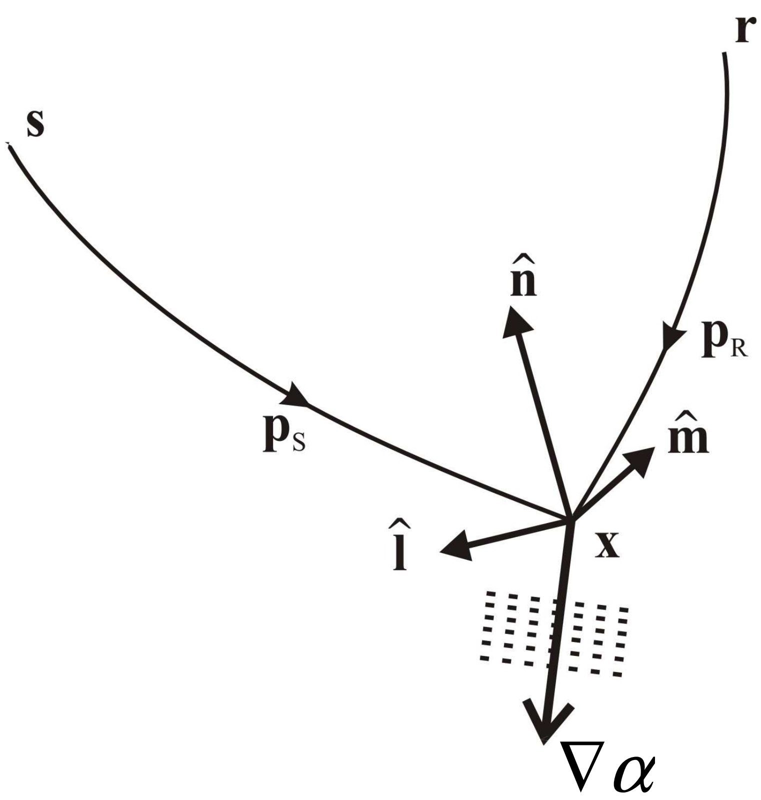
$$\gamma_A = -\mathbf{w}_R^{\text{symplectic}}\mathbf{w}_S' = -\mathrm{v}_\mathrm{R}\frac{\mathrm{d}\,P_\mathrm{S}}{\mathrm{d}z} - P_\mathrm{R}\frac{\mathrm{d}\,\mathrm{v}_\mathrm{S}}{\mathrm{d}z} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\rho}{q_\alpha}\right)$$

cf. generalized Born approximate kernel

$$\mathbf{K}^E(\mathbf{x},\mathbf{r},\mathbf{s}) = -\underline{\mathbf{v}}_\mathrm{R}^{(0)\mathrm{T}}\nabla\underline{P}_\mathrm{S}^{(0)} + \underline{P}_\mathrm{R}^{(0)\mathrm{T}}\nabla.\underline{\mathbf{v}}_\mathrm{S}^{(0)}$$

Schlumberger

Generalized Born *v.* 1D Bremmer



at reflection point use Snell coordinates

$$\hat{\mathbf{n}} = -\operatorname{sgn}(\mathbf{p}_R + \mathbf{p}_S)$$

$$\hat{\mathbf{m}} = \operatorname{sgn}(\hat{\mathbf{n}} \times \mathbf{p}_S)$$

$$\hat{\mathbf{l}} = \hat{\mathbf{m}} \times \hat{\mathbf{n}}$$

so

$$\hat{\mathbf{l}} \cdot (\mathbf{p}_R + \mathbf{p}_S) = 0$$

Generalized Born *v.* 1D Bremmer

$$\mathbf{K}^{E}(\mathbf{x},\mathbf{r},\mathbf{s})=\underline{\mathbf{v}}_{\mathrm{R}}^{(0)\mathrm{T}}\frac{\partial\underline{\mathbf{t}}_{\mathrm{S}\,j}^{(0)}}{\partial x_{j}}-\underline{\mathbf{t}}_{\mathrm{R}\,j}^{(0)\mathrm{T}}\frac{\partial\underline{\mathbf{v}}_{\mathrm{S}}^{(0)}}{\partial x_{j}}$$

$$\approx-\mathbf{g}(\mathbf{r})\,\mathscr{T}_{\mathrm{R}}\;\Gamma^{E}\;\mathscr{T}_{\mathrm{S}}\;\mathbf{g}^{\mathrm{T}}(\mathbf{s})$$

+ source/receiver directivity derivatives

in local Snell coordinates

$$\mathbf{K}^{E}(\mathbf{x},\mathbf{r},\mathbf{s})\approx\underline{\mathbf{v}}_{\mathrm{R}}^{(0)\mathrm{T}}\frac{\partial\underline{\mathbf{t}}_{\mathrm{S}n}^{(0)}}{\partial n}-\underline{\mathbf{t}}_{\mathrm{R}n}^{(0)\mathrm{T}}\frac{\partial\underline{\mathbf{v}}_{\mathrm{S}}^{(0)}}{\partial n}$$

+ transverse terms

Generalized Born *v.* 1D Bremmer

$$\begin{pmatrix} \underline{\mathbf{v}} \\ -\underline{\mathbf{t}}_j \end{pmatrix}_{\mathrm{R}}^{(0)} = \mathscr{T}(\mathbf{x},\mathbf{r}) \begin{pmatrix} \mathbf{g} \\ \mathbf{t}_j \end{pmatrix}(\mathbf{x})\ \mathbf{g}^{\mathrm{T}}(\mathbf{r})$$

ray amplitude coefficients

$\mathbf{g}, \mathbf{t}_j$ normalized for energy-flux along ray

$$\begin{pmatrix} \underline{\mathbf{v}} \\ -\underline{\mathbf{t}}_n \end{pmatrix}_{\mathrm{R}}^{(0)} = \tilde{\mathscr{T}}(\mathbf{x},\mathbf{r}) \begin{pmatrix} \tilde{\mathbf{g}} \\ \tilde{\mathbf{t}}_n \end{pmatrix}(\mathbf{x})\ \mathbf{g}^{\mathrm{T}}(\mathbf{r})$$

$$\tilde{\mathbf{g}} = \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{V}}_{\mathrm{R}}\right)^{-1/2} \mathbf{g}$$

normalized for energy-flux in $\hat{\mathbf{n}}$ direction

$$\tilde{\mathscr{T}} = \mathscr{T} \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{V}}_{\mathrm{R}}\right)^{1/2}$$

Schlumberger

Generalized Born *v.* 1D Bremmer

$$\mathbf{K}^{E}(\mathbf{x},\mathbf{r},\mathbf{s}) \approx \underline{\mathbf{v}}_{\mathrm{R}}^{(0)\mathrm{T}} \frac{\partial \underline{\mathbf{t}}_{\mathrm{S}n}^{(0)}}{\partial n} - \underline{\mathbf{t}}_{\mathrm{R}n}^{(0)\mathrm{T}} \frac{\partial \underline{\mathbf{v}}_{\mathrm{S}}^{(0)}}{\partial n}$$

+ transverse terms

$$\approx -\mathbf{g}(\mathbf{r})\, \tilde{\mathscr{T}}_{\mathrm{R}}\ \gamma\, \tilde{\mathscr{T}}_{\mathrm{S}}\ \mathbf{g}^{\mathrm{T}}(\mathbf{s})$$

+ source/receiver directivity derivatives
+ transverse terms
+ derivatives of $\tilde{\mathscr{T}}$

where

$$\gamma = -\begin{pmatrix} \tilde{\mathbf{t}}_{\mathrm{R}\,n}^{\mathrm{T}} & \tilde{\mathbf{v}}_{\mathrm{R}}^{\mathrm{T}} \end{pmatrix} \frac{\partial}{\partial n} \begin{pmatrix} \tilde{\mathbf{v}}_{\mathrm{S}} \\ \tilde{\mathbf{t}}_{\mathrm{S}\,n} \end{pmatrix}$$

generalized Born

$$= -\,\mathbf{w}_{R}^{\mathrm{symplectic}} \mathbf{w}'_{S}$$

1D Bremmer

Schlumberger

Conclusions – Perturbation Born

- standard perturbation Born provides the Fréchet differential kernel
- standard perturbation Born scattering theory is suitable for small, isolated perturbations
- it describes the travel-time differential kernel, which is zero on the ray path

Conclusions – Error Born

- generalized error Born scattering theory is needed for extended scatterers
- generalized Born models reflections from gradients as in the 1D Bremmer method
- forms foundation for inverse theory *a la* 1D exact inverse methods
  - invert for parameter gradients
  - proceed as generalized Radon transform inversion except inverting for parameter gradients not parameters, with *curl-free* constraint

    (R. Burridge, M.V. de Hoop, D. Miller and C. Spencer, 1998.

    Multiparameter inversion in anisotropic media,

    Geophys. J. Int., **134**, 757-777)