Correlation structures corresponding to forward rates

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1. Introduction

The theory of derivative pricing can maybe be described as the successive intents to mimic Black-Scholes' line of argumentation in different contexts.

When we try to extend their analysis to the interest rates (or commodity) markets we quickly realize that models driven by a sole source of uncertainty are insufficient. The reason is that the natural object to model in this case is made up of potentially infinitely many (correlated) points.

Therefore, modelling the evolution of interest rates requires the generalization of Black-Scholes' approach to a multi-factor setting. In doing this, many different extensions have been proposed, but perhaps the most celebrated one is the one proposed by Heath, Jarrow and Morton. Their approach is based on the idea that forward rates could be taken as state variables in the model, as opposed to be quantities derived from the spot rate.

A key issue affecting the performance of these type of interest rate models is the number and identity of the factors employed to describe the dynamic evolution of the term structure. A technique that has proven quite useful to this end is principal components analysis (PCA). PCA allows us to find the most important of the explanatory (1-dimensional) variables by diagonalizing the correlation matrix of interest rate changes. This was first done by R. Litterman and J. Scheinkman [11] who found that three factors, identified as *level*, *slope* and *curvature*, are enough to explain a big part of the variance present in the treasury yield curve. This property appears to be true also in other interest rate markets.

An interesting question is what are the main factors in the context of a curve of commodity futures. It turns out that the empirical results due to R. Litterman and J. Scheinkman in the case of the treasury curve are also true in the case of commodity futures. For example G. Cortazar and E. Schwartz [4] report a very similar structure in the case of copper.

It is somewhat surprising that the results of applying PCA seem to be the same no matter whether we are working in the context of interest rates or in the context of commodities. This fact would seem to indicate that the different empirical correlation matrices computed from real data share some *fundamental* structure.

Given that spot rates for different tenors are quantities that share a fair amount of information, could this be a statistical artifact?

The answer appears to be yes. I. Lekkos [10] performed the same type of analysis to forward rates instead of spot rates and found that the spectral structure is far "weaker".

In a recent paper L. Forzani and C. Tolmasky [5] showed that the observed spectral decomposition of the correlation matrices corresponding to spot rates can be recovered if we assume that the correlation structure is driven by a 1-dimensional model.

A few questions arise from this:

1) Is there any clear pattern in the spectral structure of the correlation matrix of forward rates?

2) What is the forward structure implied in the model proposed by Forzani and Tolmasky ? Is that a reasonable structure?

3) Can we find a parametric model describing the spectral structure of the correlation matrix corresponding to forward rates?

2. Heath-Jarrow-Morton's Equation

The starting point in derivatives pricing is Black-Scholes' model. In this, a stock S is assumed to evolve following the following stochastic differential equation:

$$\frac{dS}{S} = \mu S + \sigma S dW \tag{2.1}$$

where W is a Brownian Motion and μ, σ are constants representing the expected return and volatility corresponding to S.

In the attempt to adapt this framework to work for interest rates we can try to model the short rate and derive the rest from it. The resulting rates will be perfectly correlated creating problem in cases in which non-perfect correlation. Adding factors appeared to be an alternative, but it is not exactly clear how to do this for a general number of factors. Heath, Jarrow and Morton proposed to model the forward rate curve directly:

$$\frac{df(t,T)}{f(t,T)} = \alpha(s,T)ds + \sum_{i=1}^{n} \sigma_i(t,T)dW_i \tag{2.2}$$

where $f(t,T)$ is the instantaneous forward at time T observed at time t. In principle we would need one Brownian Motion per forward but, given that the correlation of different rates could is positive we could try to reduce dimensionality. Also, to determine the model, we need to choose the σ's (the α is determined by precluding arbitrage). To this end we use principal components analysis.

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