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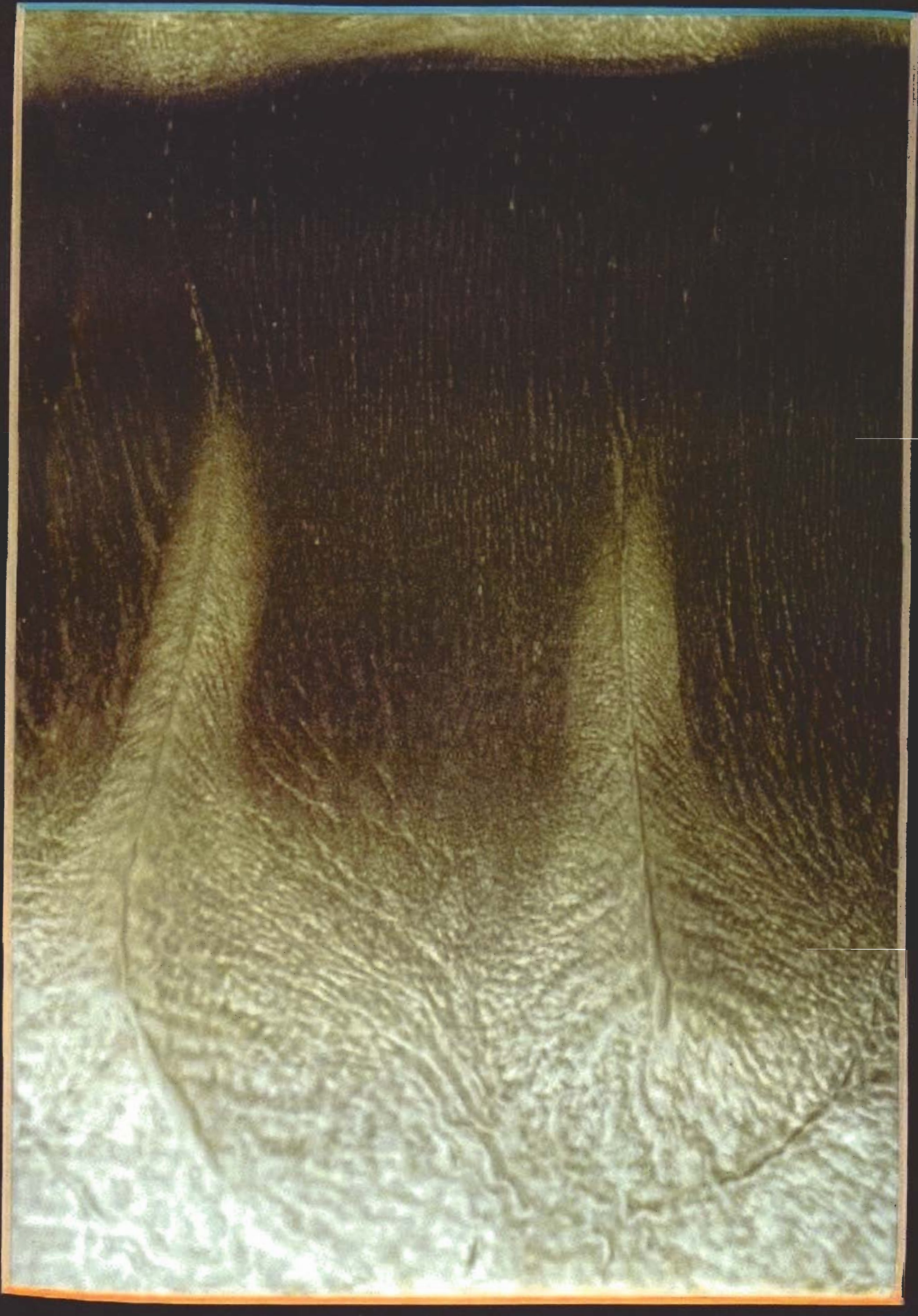
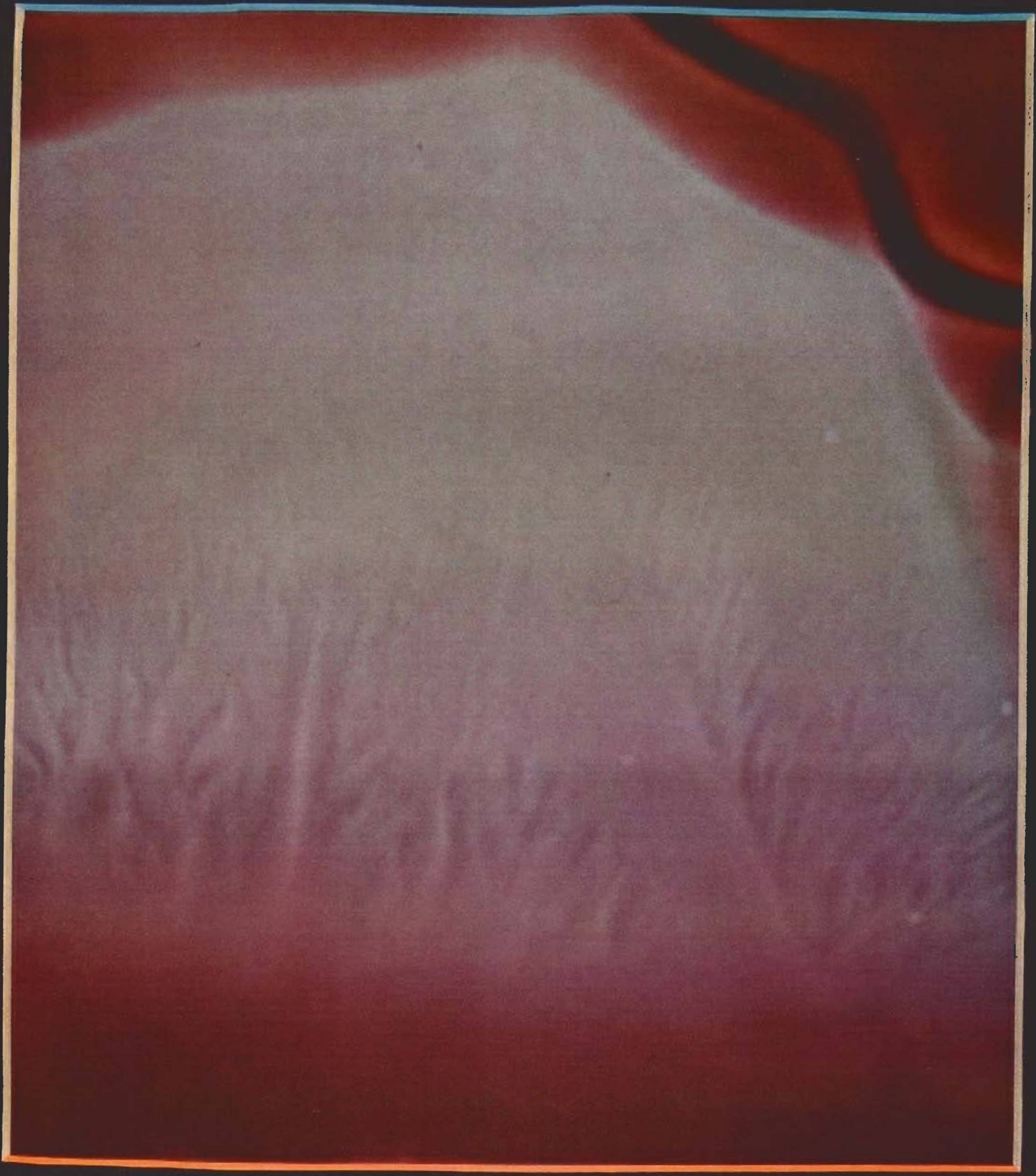
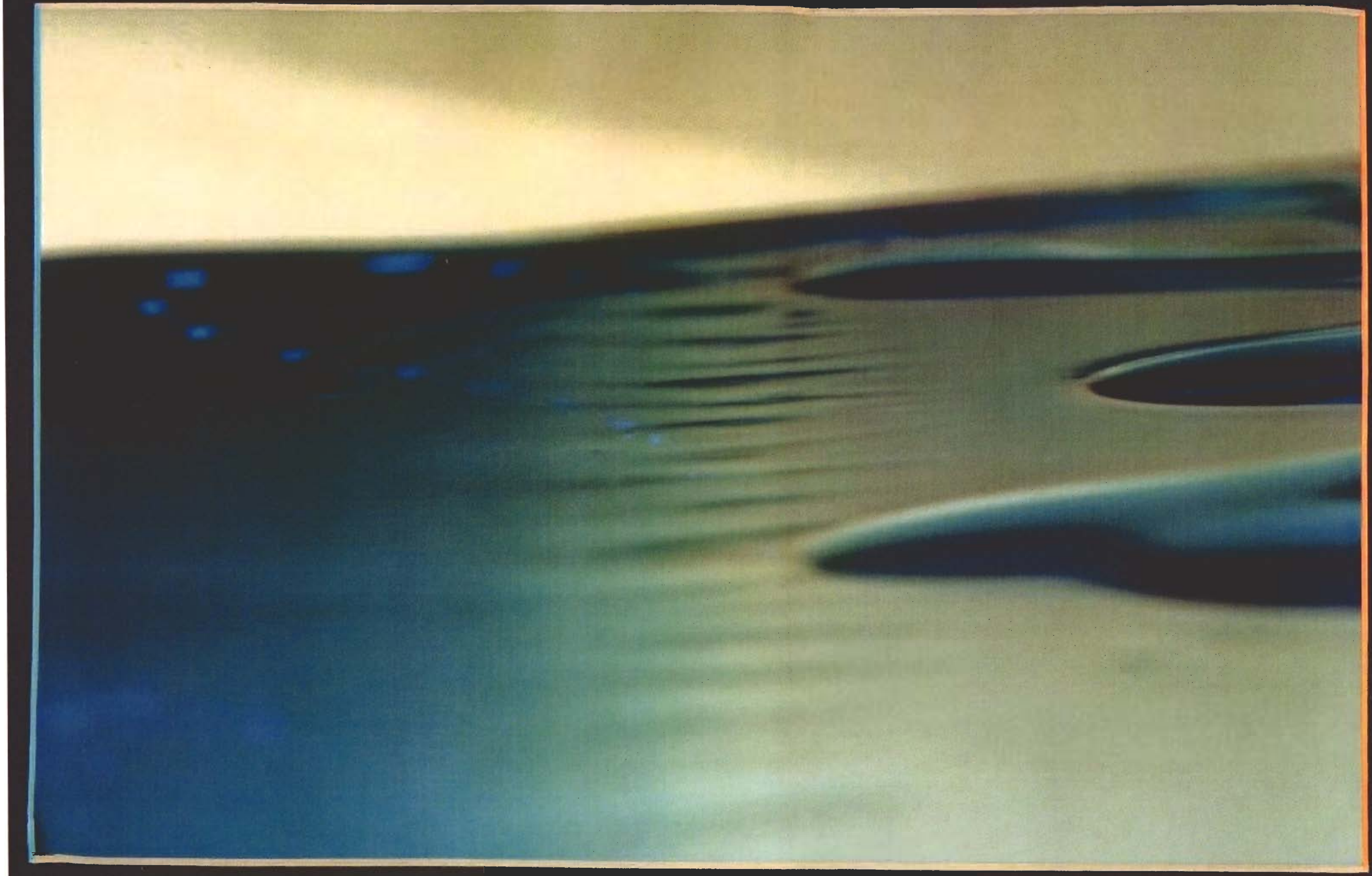
The Richness of Thin Films

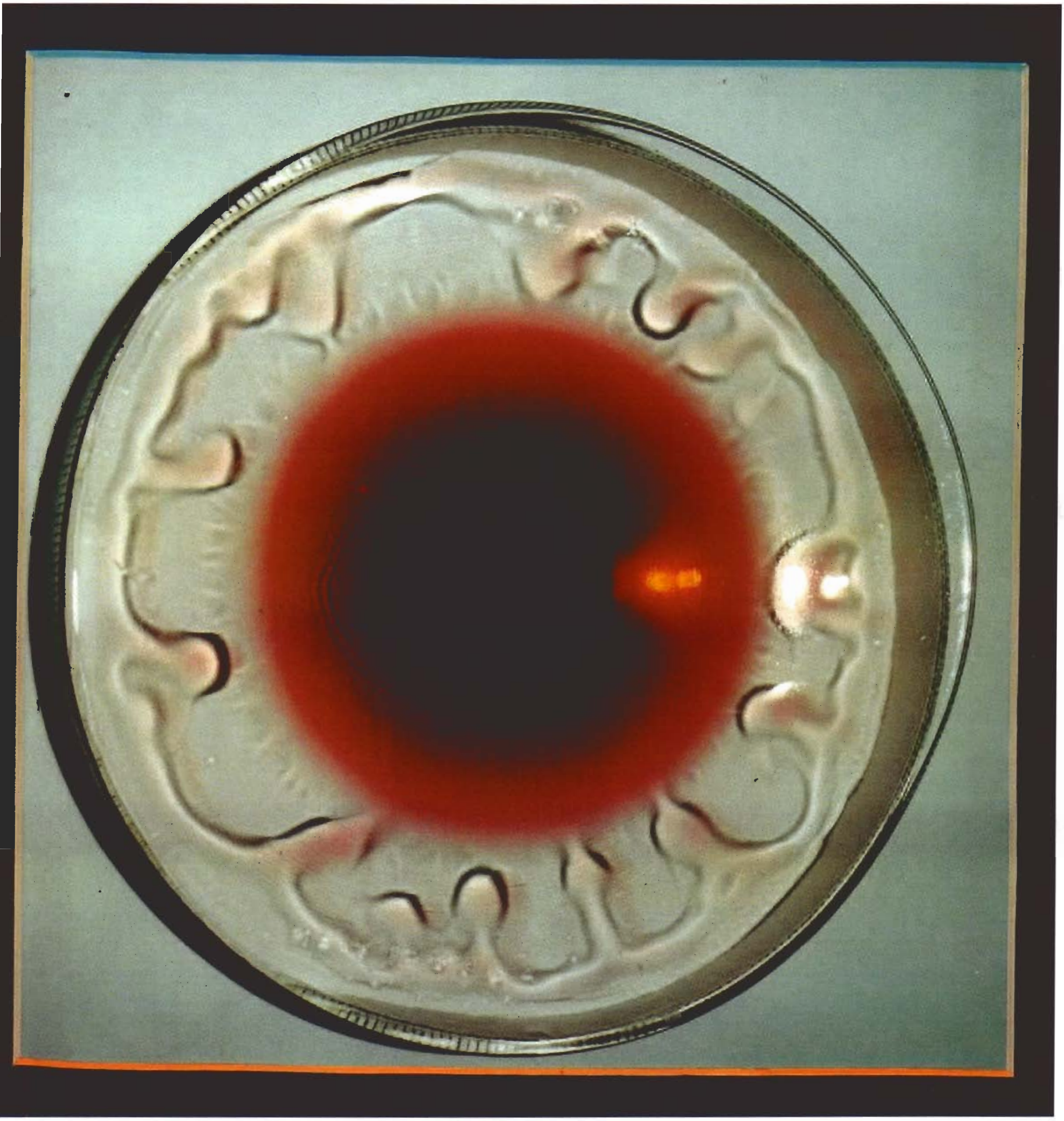
I will present a survey of modelling, computational, and analytical work on thin liquid films of viscous fluids. I will particularly focus on films that are being acted on by more than one force. For example, if you've painted the ceiling, how do you model the effects of surface tension and gravity? How do you study the dynamics of the air/liquid interface? How do things change if you're considering a freshly painted wall? Or floor?

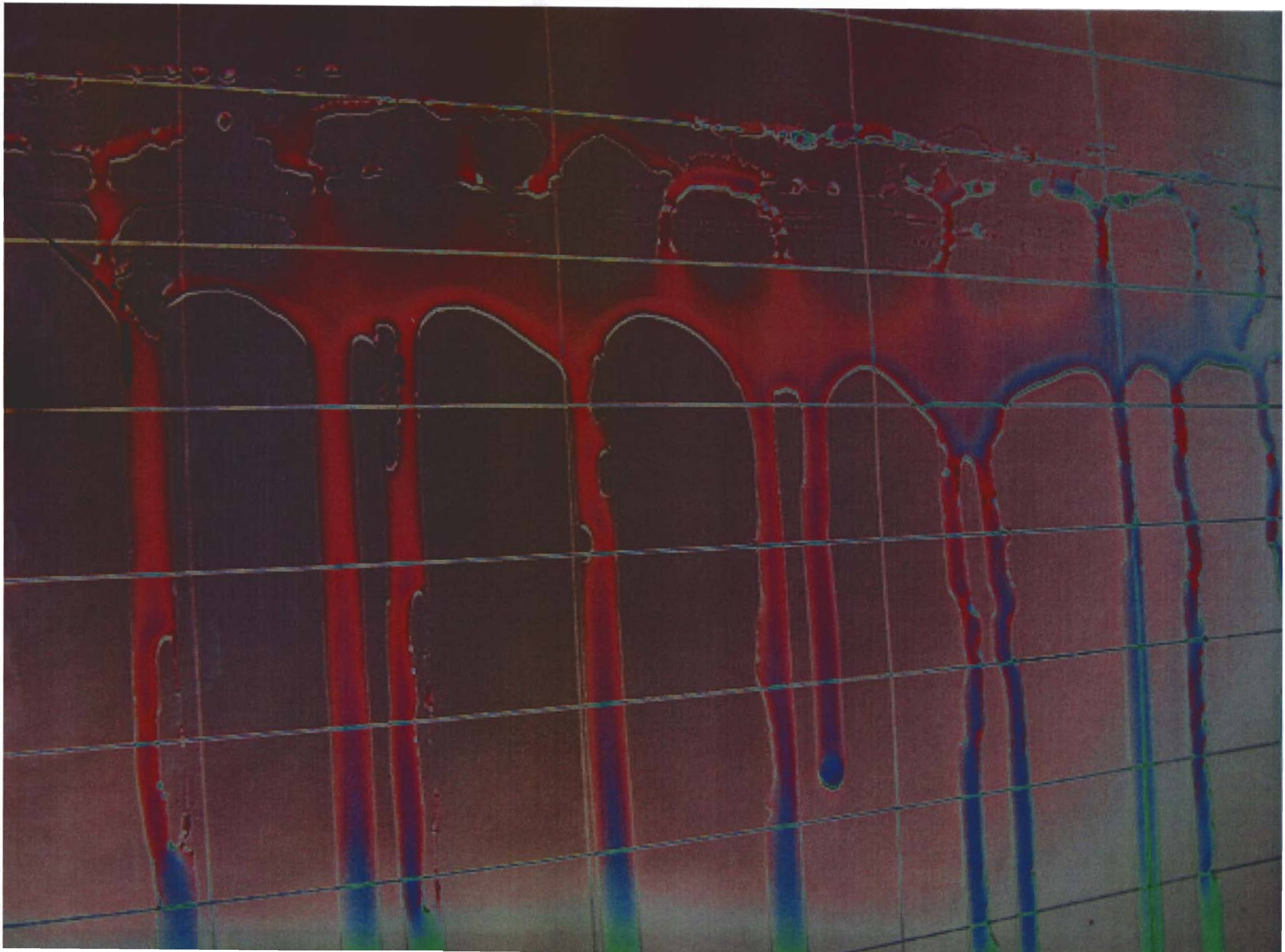
The talk will be aimed at a general audience. In particular, I hope that students will attend.



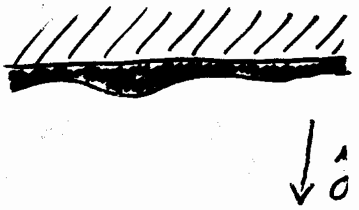








$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x \qquad f, g \geq 0$$



gravity-destabilized film

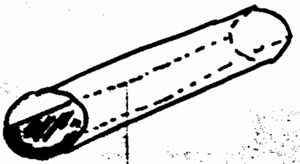
$$h_t = -(h^3 h_{xxx})_x - B(h^3 h_x)_x$$

Ehrhardt + Davis

net repulsive van der Waals forces

$$h_t = -(h^3 h_{xxx})_x - B(\tfrac{1}{h} h_x)_x$$

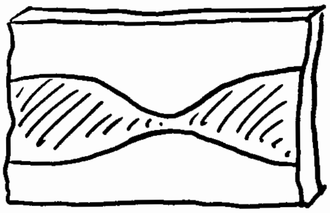
Williams + Davis



flow in a pipe

$$h_t = -(h^3 h_{xxx})_x - B(h^3 h_x)_x$$

Hammond
Jensen



gravity-destabled Hele-Shaw cell

$$h_t = -(h h_{xxx})_x - B(h h_x)_x$$

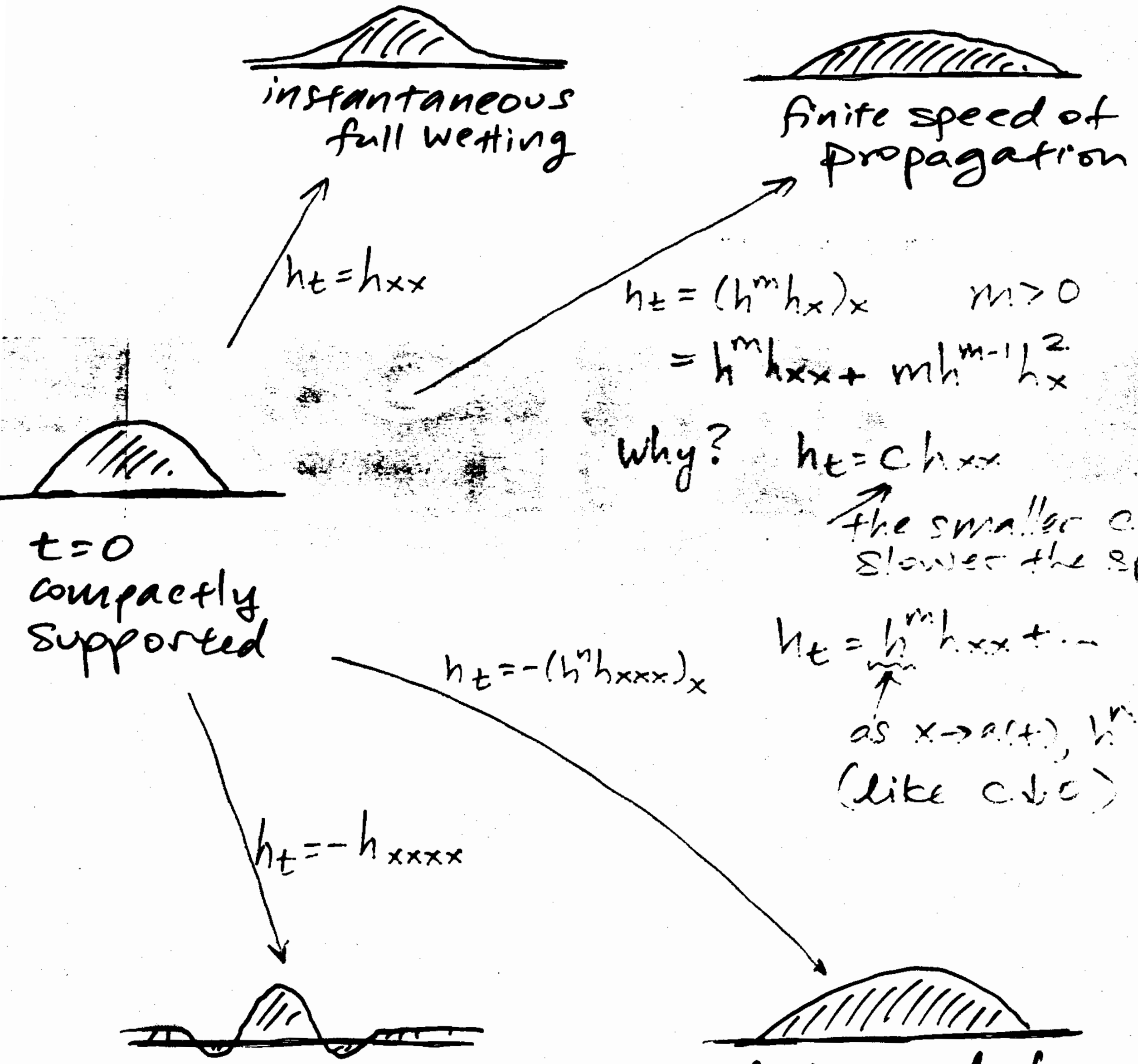
Goldstein + Pesci + Shelley

Aphid Population Dynamics

$$h_t = -(h h_{xxx})_x - ((h-c)h_x)_x$$

Lewis

Mathematical Issues concerning $h_t = -(h^n h_{xxx})_x$



instantaneous full wetting

$h_t = h_{xx}$

finite speed of propagation

$h_t = (h^m h_x)_x \quad m > 0$

$= h^m h_{xx} + m h^{m-1} h_x^2$

why? $h_t = c h_{xx}$

the smaller c is, the slower the spread.

$h_t = \underset{\uparrow}{\underbrace{h^m}} h_{xx} + \cdots$

as $x \to a(t)$, $h^m \downarrow 0$

(like $c \downarrow 0$)

t=0

compactly supported

$h_t = -(h^n h_{xxx})_x$

$h_t = -h_{xxxx}$

infinite speed of propagation.

<u>AND</u> fluid penetrates the surface :(

finite speed of propag.

$h(x,t) \geq 0 \quad \forall x, t$

Bernis + Friedman '91

Bertozzi + Pugh

Baretta, Bertsch, dal Passo } '95

Bernis '96

Why is there finite speed of propagation?
Why is nonnegativity preserved?

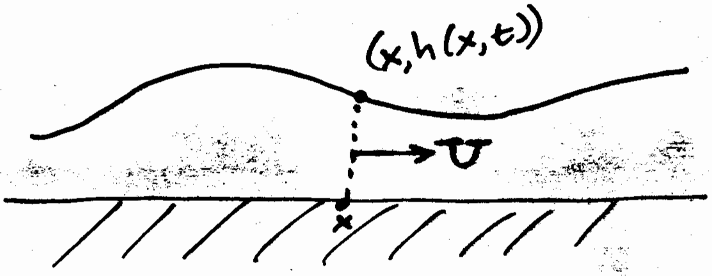
The PDEs $h_t = -(h^n h_{xxx})_x$

$h_t = -(h^n h_{xxx})_x - (h^m h_x)_x$

have the form

$$\boxed{h_t + (hU)_x = 0}$$

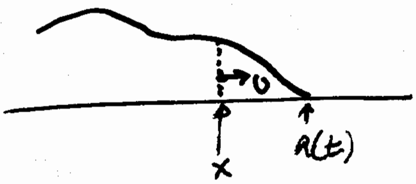
This arises from conservation of mass.



U = vertically averaged fluid velocity above position x at time t.

$= h^{n-1} h_{xxx}$ (or $h^{n-1}h_{xxx} + h^{m-1}h_x$ or...)

Finite Speed of propagation?



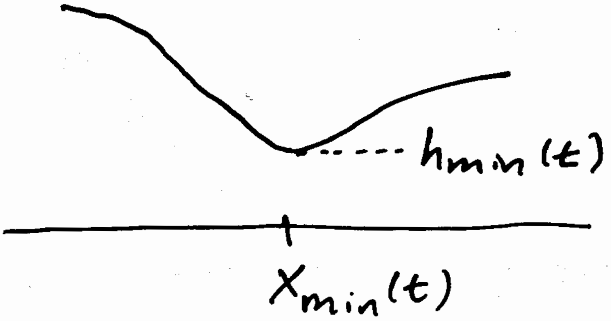
contact line is at $x = a(t)$.

For it to move, expect:

$$\frac{d}{dt}a(t) = \lim_{x \to a(t)} U(x,t) = \lim_{x \to a(t)} h^{n-1}(x,t)h_{xxx}(x,t)$$

Since $h \downarrow 0$ as $x \to a(t)$, need $h_{xxx} \uparrow \infty$ like h^{1-n}... delicate!!

How does the equation "notice" if the solution begins to form a dry patch?



Can $h_{min}(t) \downarrow 0$?

$$h_{min}(t) = h(x_{min}(t), t)$$

$$\frac{d}{dt} h_{min}(t) = \frac{\partial h}{\partial x}(x_{min}(t),t) \cdot \frac{d}{dt} x_{min} + \frac{\partial h}{\partial t}(x_{min}(t),t)$$

$$= \frac{\partial h}{\partial t}(x_{min}(t),t) \qquad \text{since } \frac{\partial h}{\partial x} = 0 \text{ at } x_{min}$$

$$= -h \frac{\partial U}{\partial x}\Big|_{x_{min}(t)} - \frac{\partial h}{\partial x} U \Big|_{x_{min}(t)}$$

$$= -h \frac{\partial U}{\partial x}\Big|_{x_{min}(t)} = -h_{min}(t)\, U_x(x_{min}(t),t)$$

$\Rightarrow$

$$h_{min}(t) = h_{min}(0)\, e^{-\int_0^t \frac{\partial U}{\partial x}(x_{min}(s),s)\,ds}$$

For $h_{min} \downarrow 0$ need $U_x \uparrow \infty$. Recall

$U = h^{n-1} h_{xxx}$ (or something analogous...)

$\Rightarrow$ loss of regularity.

$h(x,0) = h_0(x) \geq 0$ ← nonnegative initial data.

Q: How do you construct nonnegative solutions?

A: Very carefully!

idea 1: regularize the equation...

Let $\varepsilon > 0$ be a small parameter.

Find the solution h_ε of the initial value problem

$$\circledast \begin{cases} h_{\varepsilon t} = -\left(\left(h_\varepsilon^n + \varepsilon\right) h_{\varepsilon xxx}\right)_x \\ h_\varepsilon(x,0) = h_0 \quad \text{at } t=0 \end{cases}$$

then take $\varepsilon \downarrow 0$.

Hmmm... $\circledast$ won't have nonnegative solutions in general. And even if you can prove the limit is nonnegative (you can*) then this regularization will not be useful computationally.

*Bernis + Friedman, '91

Idea #2 (Bernis + Friedman, '91)

Trade one degeneracy for another.

Let h_ε be a solution of

$$* \begin{cases} h_{\varepsilon t} = -(f_\varepsilon(h_\varepsilon)\, h_{\varepsilon xxx})_x \\ h_\varepsilon = h_0 + \varepsilon^{1/5} & t=0 \end{cases}$$

where $f_\varepsilon(y) = \dfrac{y^{n+4}}{\varepsilon y^n + y^4}$.

Then $h_\varepsilon > 0$ for all time and $h_0 = \lim h_\varepsilon$ is a weak solution of the original initial value problem.

Why that f_ε? ans: for $\varepsilon = 0$, $f_\varepsilon(y) = y^n$ ✓

$\varepsilon \neq 0$ then $f_\varepsilon(y) \sim y^4$ for $y \ll 1$.

How is $h_t = -(h^4 h_{xxx})_x$ with positive initial data going to preserve positivity?

Consider $h_t = -(h^n h_{xxx})_x$ with periodic or Neumann Boundary Conditions?

Dissipated Energy

$$\frac{d}{dt} \frac{1}{2} \int h_x^2 dx = - \int h^n h_{xxx} dx \leq 0$$

$$\Rightarrow \int h_x^2(x,t) dx \leq \int h_{0x}^2(x) dx$$

$\Rightarrow$ if $\int h_{0x}^2 dx < \infty$ then $h \in H^1$ for each time.

$\Rightarrow h \in L^\infty$ and $C^{1/2}$

Dissipated "Entropy"

$$\frac{d}{dt} \int h^{2-n} = -(n-1)(n-2) \int h_{xx}^2$$

$$\Rightarrow \text{if } n=4 \quad \frac{d}{dt} \int \frac{1}{h^2} = -6 \int h_{xx}^2 \leq 0$$

$$\Rightarrow \text{if } n=4 \quad \int \frac{1}{h^2} \leq \int \frac{1}{h_0^2}$$

$\Rightarrow$ if $\int \frac{1}{h_0^2} < \infty$ then $\int \frac{1}{h^2} < \infty$ at each time

Conclude

If $n=4$ and $h_0>0$ then the solution of

$$h_t = -(h^n h_{xxx})_x$$

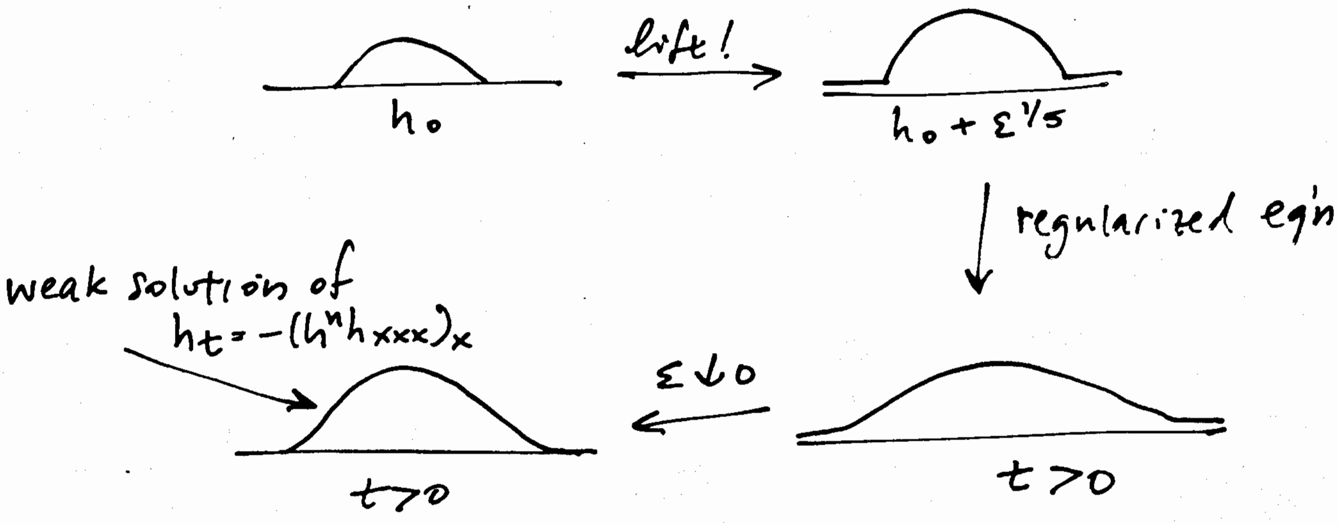
will have finite $\int \frac{1}{h^2}$ at each time and will be $C^{1/2}$ at each time.

Fact: if $h_{min}(t) \downarrow 0$ at $x_{min}(t)$ and h remains $C^{1/2}$ then this will force

$$\int \frac{1}{h^2} \to \infty.$$

Conclude: $h>0$ for all time.

regularization



Consider

$$h_t = -\underset{\substack{\uparrow \\ \text{linearly} \\ \text{stabilizing} \\ \text{term}}}{(f(h)h_{xxx})_x} - \underset{\substack{\uparrow \\ \text{linearly} \\ \text{destabilizing} \\ \text{term}}}{(g(h)h_x)_x}$$

Q: Can the second-order term overpower the fourth-order term? This would be undesirable for a thin film... h would have left the regime of validity...

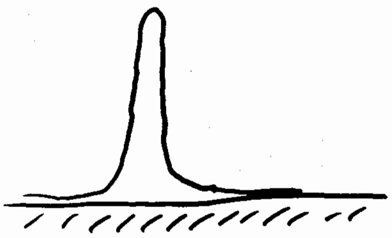
Recall $$h_t = \underset{\substack{\uparrow \\ \text{stabilizing}}}{h_{xx}} + \underset{\substack{\nwarrow \\ \text{destabilizing}}}{h^p}$$

know $\frac{dy}{dt} = y^p$ can blow up in finite time if $p > 1$.

thm: If $p > 1$ then $\exists$ initial data h_0 that yield solutions with $h \uparrow \infty$ in finite time.

critical exponent: $p = 1$

Intuition:



h_0 is tall and narrow

so tall that $h_t = h^p$ drives top to blow up faster than $h_t = h_{xx}$ can bring it down.

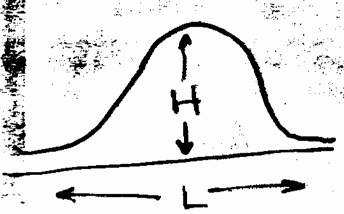
Consider $h_t = -(h^n h_{xxx})_x - B(h^m h_x)_x$

Hocherman + Rosenau (Phys. D. '93) conjectured that if $m>n$ then $\|h\|_\infty \to \infty$ in finite time.

Note: their conjecture was for a wide class of equations, including those that do not preserve the sign of the solution. Ours have $h_0 \geq 0$ at $t=0 \Rightarrow h \geq 0$ at $t>0$

Does this change the blow-up conjecture?

Modified Conjecture: (Bertozzi + P. '97)



volume conservation:

$$H \cdot L \leq V < \infty$$

(this uses that the solution conserves sign)

Find a critical exponent...

$$(h^n h_{xxx})_x \sim (h^m h_x)_x$$

$$\frac{H^{n+1}}{L^4} \sim \frac{H^{m+1}}{L^2} \Rightarrow H^{n-m} \sim L^2$$

$$\Rightarrow H^{n-m+2} \sim H^2 L^2 \leq V^2 < \infty$$

If $H \uparrow \infty$ then must have $n-m+2 \leq 0$

$$\Rightarrow \boxed{m \geq n+2} \; !$$

Bertozzi + P.

Conjecture:

consider $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$

A: $m < n+2 \Rightarrow$ positive/nonnegative solutions remain bounded for all time

$\|h(\cdot,t)\|_{L^\infty} \le C < \infty$ ↖ determined by $\mathcal{B}$, m, n, and h_0

proven
Bertozzi + P. 97

B: $m \ge n+2$ and $m \le n/2 \Rightarrow$ positive/nonnegative solutions grow at most exponentially in time $\|h(\cdot,t)\|_{L^\infty} \le Ce^{at}$

C & a det'd by $\mathcal{B}$, m, n, and h_0

proven
Bertozzi + P. 97

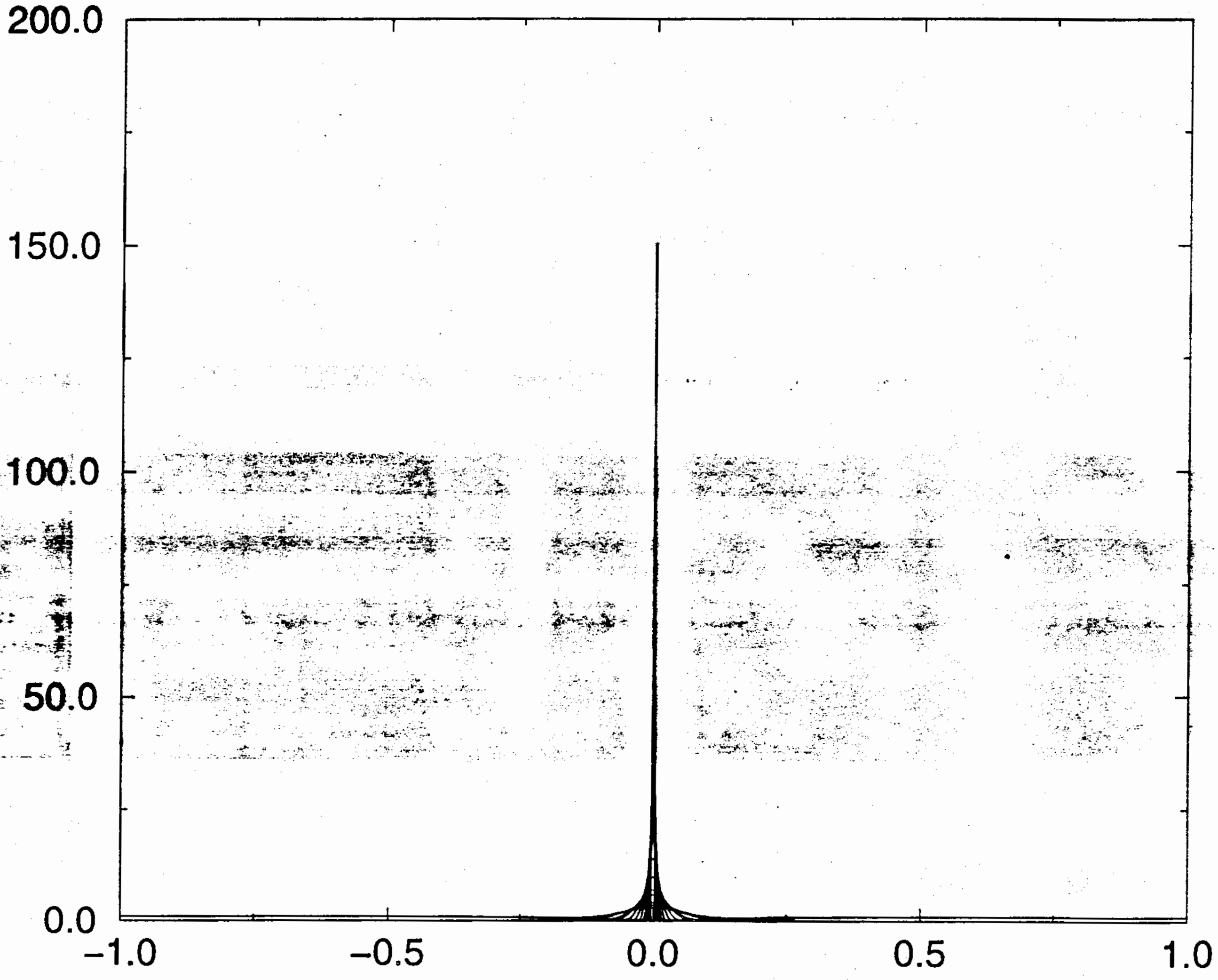
C: $m \ge n+2$ and $m > n/2 \Rightarrow \exists\, h_0$ such that $\|h(\cdot,t)\|_{H^1}$ and $\|h(\cdot,t)\|_{L^\infty} \uparrow \infty$ in finite time

num. evidence
Bertozzi + P. '97

proven for $n=1$
Bertozzi + P. '99

┌ Note: conjecture stated and A+B proven for general eqn $h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x$ ┘

Critical Case


Numerical evidence of blowup

$h_t = -(h^4 h_{xxx})_x - 20(h^7)_{xx}$ singularity

$$h_t = -(h^4 h_{xxx})_x - 140(h^6 h_x)_x$$

$n=4$ $m=6$ critical case

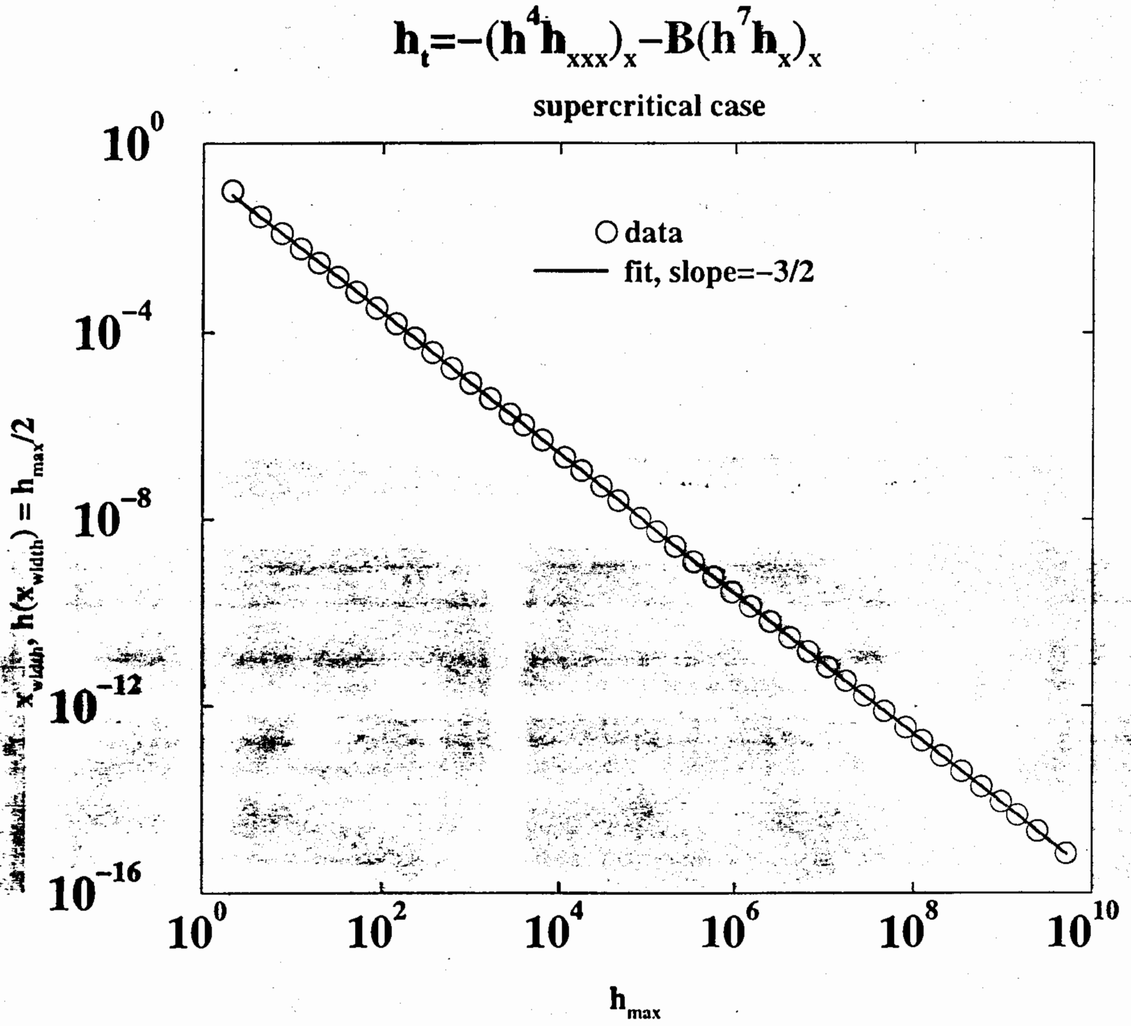


Figure 2: Blowup in supercritical case with $m = n + 3$. Here we confirm the scaling $q = (m - n)/2$ for the blowup profile.

Theorem (Bertozzi + P. '97)

If h is smooth and nonnegative and periodic and $m<n+2$ then h is uniformly bounded. Specifically, $\exists\, M$ such that

$$\|h\|_{H^1} \leq M < \infty \quad \text{for all time}$$

and hence $\|h\|_{L^\infty} \leq M < \infty$ $\forall t$. M depends on h_0, m, n, and the length of the interval.

Proof: the energy

$$\mathcal{E}(h) := \int \tfrac{1}{2} h_x^2 - G(h)\, dx$$

where $G''(h) = h^{m-n}$ is dissipated in time:

$$\frac{d}{dt}\mathcal{E}(h) \leq 0 \Rightarrow \mathcal{E}(h(\cdot,t)) \leq \mathcal{E}(h_0) < \infty$$

$\mathcal{E}$ is unsigned, but if $m<n+2$ then one can use an interpolation inequality and $\exists\, C$ such that

$$\begin{aligned}\frac{1}{4}\|h\|_{H^1} &\leq \mathcal{E}(h(\cdot,t)) + C \\ &\leq \mathcal{E}(h_0) + C = M. \quad //\end{aligned}$$

Dissipated Energy

$$\mathcal{E}(u(\cdot,t)) = \int \tfrac{1}{2}u_x^2 - \frac{1}{(m-n+1)(m-n+2)}\, u^{m-n+2}\, dx$$

subcritical case $m < n+2$

Gagliardo-Nirenberg

$\Rightarrow \exists\, C_1$ (determined by $\int u_0$)

such that

$$\|u(\cdot,t)\|_{H^1} \le \mathcal{E}(u(\cdot,t)) + C_1$$

so $\mathcal{E}(u(\cdot,t)) \downarrow -\infty$ impossible. Control of $\mathcal{E}$ implies control of $\|u\|_{H^1} \Rightarrow$ solutions exist all time.

critical case $m = n+2$

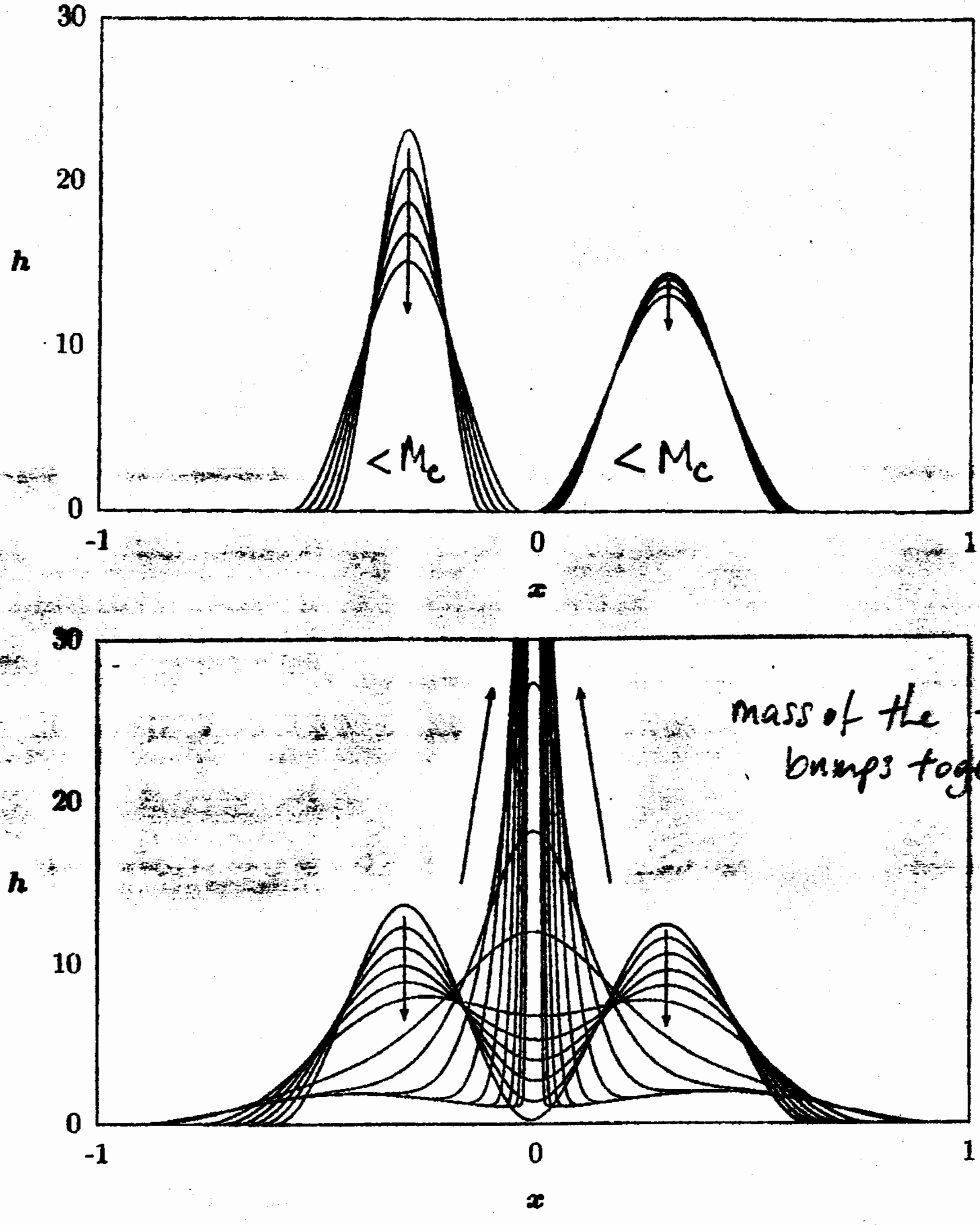
Witelski + Bernoff + Bertozzi (2003) found a sharp Sz.-Nagy inequality that yields

$$\left[\frac{1}{2} - \frac{3(\int u_0)^2}{16\pi^2}\right] \int u_x^2(x,t)\,dx \le \mathcal{E}(u(\cdot,t))$$

$\nearrow > 0$ if $\int u_0\, dx < \frac{2\sqrt{2}}{\sqrt{3}}\pi =: M_c$

Figure courtesy Witelski, Bernoff, Bertozzi (2003)

$$u_t = -(u u_{xxx})_x - (u^3 u_x)_x$$



mass of the two bumps together $> M_c$

<u>Concl:</u> if $\int u_0 < M_c$ then solution exists for all time and $\|u\|_{H^1} < C$.

<u>Q:</u> what if $\int u_0 > M_c$?

$$M_c = \frac{2\pi\sqrt{2}}{\sqrt{3}}$$ = mass of zero contact angle droplet steady state

This mass is invariant with respect to the natural rescaling.

The existence theory yields weak solutions that have zero contact angles at almost all times so we seek

self similar solutions
with compact support
& zero contact angles.

$$u(x,t) = (1+\sigma t)^{\frac{-1}{n+4}}\, U\left(\frac{x}{(1+\sigma t)^{\frac{1}{n+4}}}\right)$$

that solves $u_t = -(u^n u_{xxx}) - (u^{n+2}u_x)_x$

$\sigma = +1$ solutions are source-type. Exist for all time, spreading in self-similar manner. Requires $n<3$. The solutions have "droplet" profiles

Beretta 1997

Figure courtesy Witelski, Bernoff, Bertozzi 2003

Dynamics of a critical-case thin film equation

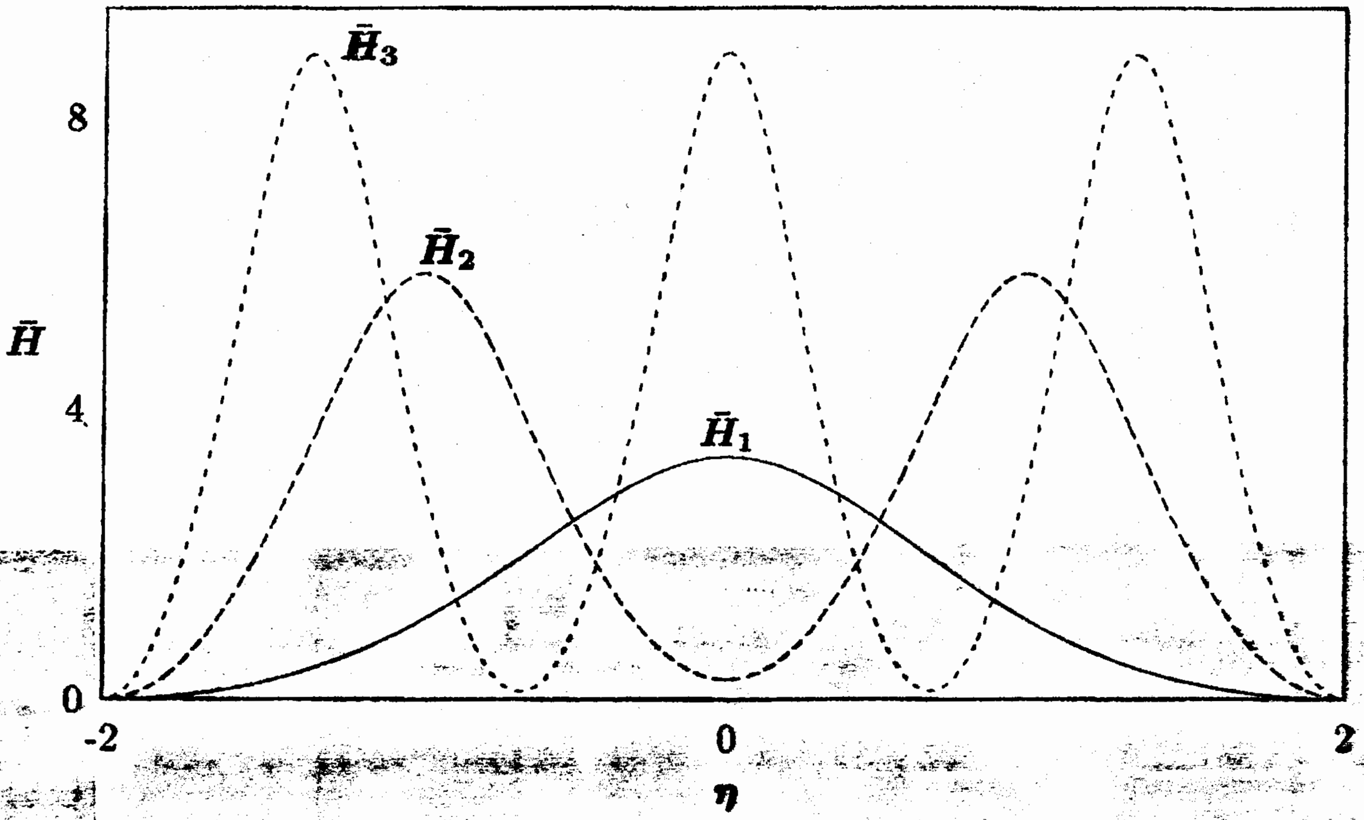


FIGURE 6. The first three blow-up similarity solutions for $\bar{L} = 2$.

$\sigma = -1$: Solutions blow up in finite time $(T=1)$ one-point focussing blow-up.

Witelski, Bernoff, Bertozzi (2003) did extensive numerics & asymptotics on $n=1$ case of equation

$$u_t = -(u u_{xxx})_x - (u^3 u_x)_x$$

They found $\exists$ countable family of compactly supported zero contact angle self-similar blow-up solutions. (One for each # of local max.)

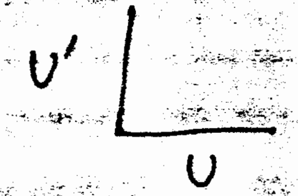
Witelski, Bernoff, Bertozzi (2003) cont.

$$u_t = -(uu_{xxx})_x - (u^3u_x)_x$$

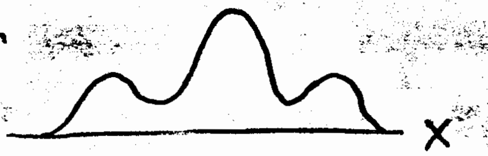
self-similar blow-up solution has profile
U that satisfies

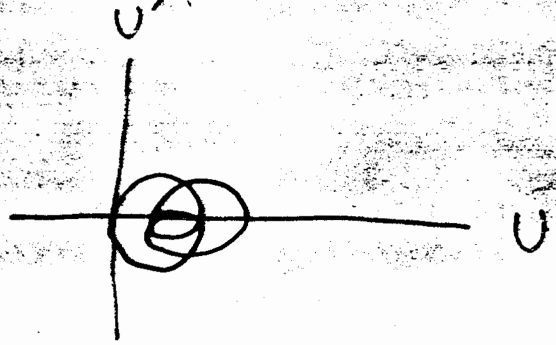
$$U''' = \frac{-1}{5}x - U^2U' \quad \circledast$$

plot a profile U in the phase plane

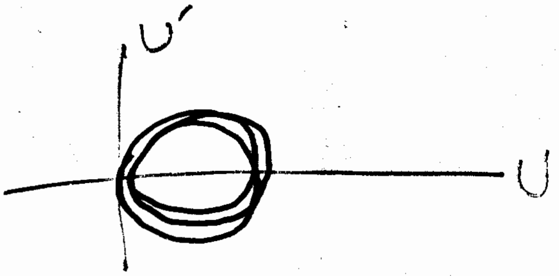


then





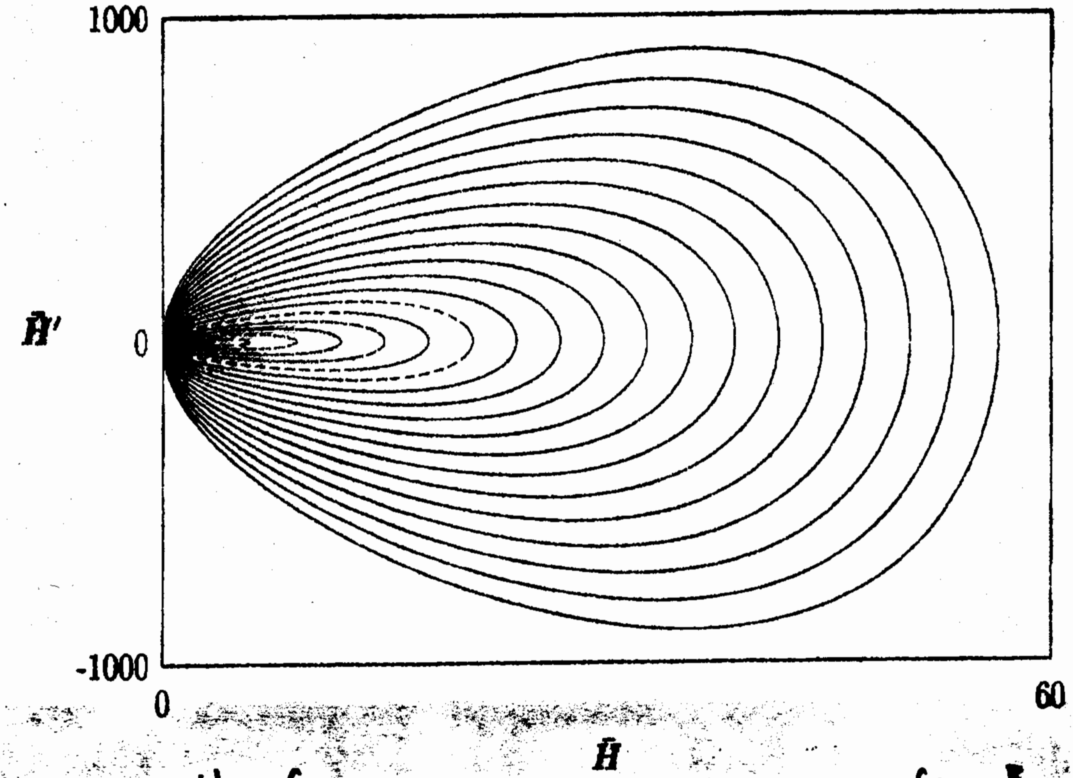
U "nearly periodic"



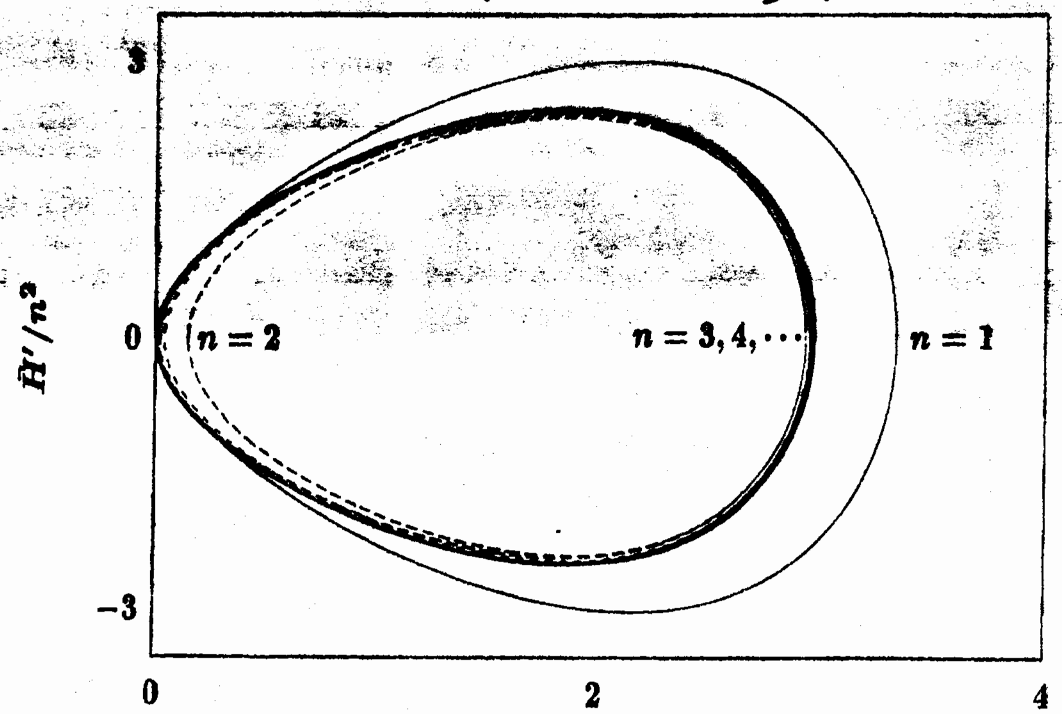
computationally observed: multibump solutions appear to be very close to steady states. and the more bumps, the larger U_{max}.

$$U(x) \to \frac{1}{n}U\left(\frac{x}{n}\right) \qquad \circledast \Rightarrow U''' = -U^2U' - \frac{1}{5}\frac{1}{n^3}x$$

Figure Courtesy Witelski, Bernoff, Bertozzi 2003



the first 20 multibump blow-up profiles } after rescaling ↓



WB^2 continued. $u_t = -(uu_{xxx})_x - (u^3u_x)_x$

$U''' = -U^2U' - \frac{1}{5}\frac{1}{n^3}x$ blow-up profile

$V''' = -V^2V'$ steady state profile

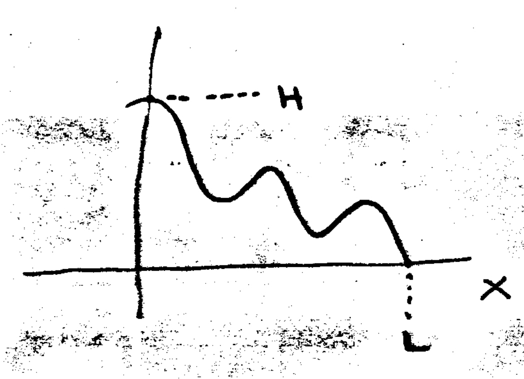
Self-similar blow-up solutions that blow up at $T=1$ satisfy the ODE

$$U^n U''' = -\frac{1}{n+4} xU - U^{n+2}U'$$

(assumed
(symmetric
about $x=c$
to integrate
up.

view this as an initial data problem:

$$\begin{cases} U''' = -\frac{1}{n+4} x U^{1-n} - U^2 U' \\ U(0) = H \\ U'(0) = 0 \\ U''(0) = \gamma \end{cases}$$



take solution until $x=L$ where $U(L)=0$. reflect about $x=0$, this is the desired profile $\mathcal{U}$.

Given H, seek γ so that the resulting solution $\mathcal{U}$ has zero contact angles.

γ_1 yields 

γ_2 yields 

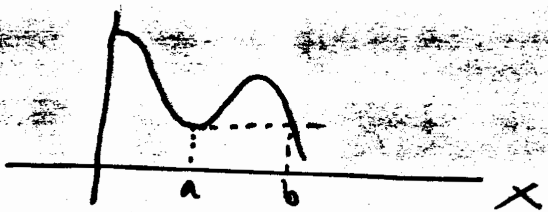
γ_3 yields..

Qualitative Properties of self-similar blowup solutions (Slepcev, Pugh 200

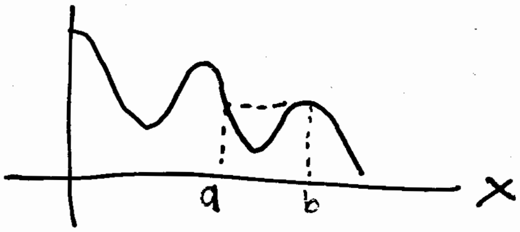
<u>theorem:</u> if $n \geq 3/2$ then $\nexists$ solutions with zero contact angles.

$3/2$ pops out of local asymptotics at contact line.

<u>theorem:</u> if U is a solution of self-similar profile equation and $0 \leq a < b$ with $U(a) = U(b)$ and $U'(a) = 0$ then $U'(b) < 0$

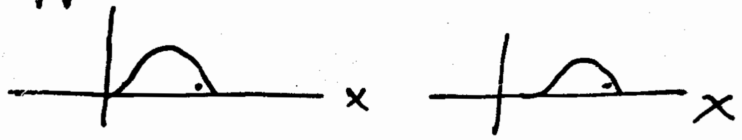


if $U'(b) = 0$ then $U'(a) < 0$



<u>corr:</u> as x increases, local maxima decrease in height. As x increases, local minima decrease in height

<u>corr:</u> Any zero contact angle solution must have support $[a, b]$ where $a < 0 < b$.



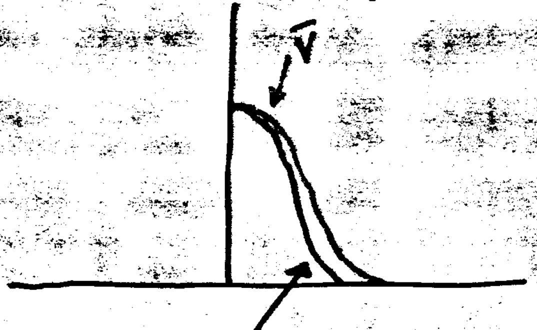
nonzero contact angles to right ⇒ inadmissible!

theorem: Any solution that blows up in finite time in a self similar manner, with zero contact angles has

$$\int u_0(x)\,dx > M_c = 2\pi\sqrt{2/3}$$

Proof: (sketch) by comparison to steady state solution

$\bar{V}$ of $\begin{cases} V''' = -V^2 V' \\ V(0) = H \\ V'(0) = 0 \\ V''(0) = -H^{3/4} \end{cases}$



U with same initial data
$U(0) = H,\ U'(0) = 0,\ U''(0) = -H^{3/4}$
has nonzero contact angle.

can prove comparison theorem: if U has zero contact angles then $U(x) > \bar{V}(x)$ at all $x \neq 0$

$$\Rightarrow \quad \int u_0 = \int U > \int \bar{V} = 2\pi\sqrt{\frac{2}{3}}$$

Note: these comparison methods show that any source-type spreading solution must have mass $< 2\pi\sqrt{\frac{2}{3}}$

Existence Results

theorem: If $0<n<3/2$, $\exists H_n$ such that if $H \leq H_n$ then $\nexists$ self similar profile U with

- $U(0)=H$ $U'(0)=0$ $U''(0)=8$
- zero contact angles
- compact support

that yields a solution $u(x,t)$ that blows up self-similarly as $t \uparrow 1$. As $n \uparrow 3/2$, $H_n \uparrow \infty$.

Note: This is the analogue of the BW^2 observation in the $n=1$ case that $\nexists$ solutions with length $> L_1$. Lengths $[0, L_1]$ correspond to $[H_{min}, \infty)$.

theorem: If $0<n<3/2$, and $k \in \mathbb{N}$, $\exists \tilde{H}_{n,k}$ such that if $H \geq \tilde{H}_{n,k}$ then $\exists$ self similar profile U with

- $U(0)=H$ $U'(0)=0$ $U''(0)=8$
- zero contact angles
- compact support

that yields a solution $u(x,t)$ with k local maxima that blows up self similarly as $t \uparrow 1$. As $n \uparrow 3/2$, $\tilde{H}_{n,k} \uparrow \infty$.
As $k \uparrow \infty$, $\tilde{H}_{n,k} \uparrow \infty$.

Shape theorem

$0<n<3/2$ and $H>\tilde{H}_{n,1}$. Let U be the self similar profile with one local maxima, zero contact angles, compact support.

then $\|U-\bar{V}\| \to 0$ as $H \uparrow \infty$.

(Recall $\bar{V}$ is the steady state droplet solution with zero contact angles and mass $2\pi\sqrt{2/3}=M_c$)

transl: if you look at a steady droplet and a droplet blow-up solution, they look very similar.

transl: steady droplets are unstable to perturbations that add small mass. Since an arbitrarily small addition of mass to $\bar{V}$ could result in initial data $u_0=U$ which will then blow up in finite time $T=1$.

Methods of existence proofs.

Good news: a shooting method works.
Bad news: the equation does not have the nice properties that

$$u_t = -(u^n u_{xxx})_x$$

$$u_t = -(u^n u_{xxx})_x \pm (u^{n+2} u_x)_x$$

have when seeking source-type solutions.

Idea Inspired By Simulations: The self similar blow-up profile U is "very close" to steady state solution V with same initial data. Let's try to slave U to V and then transfer understanding of V to U.

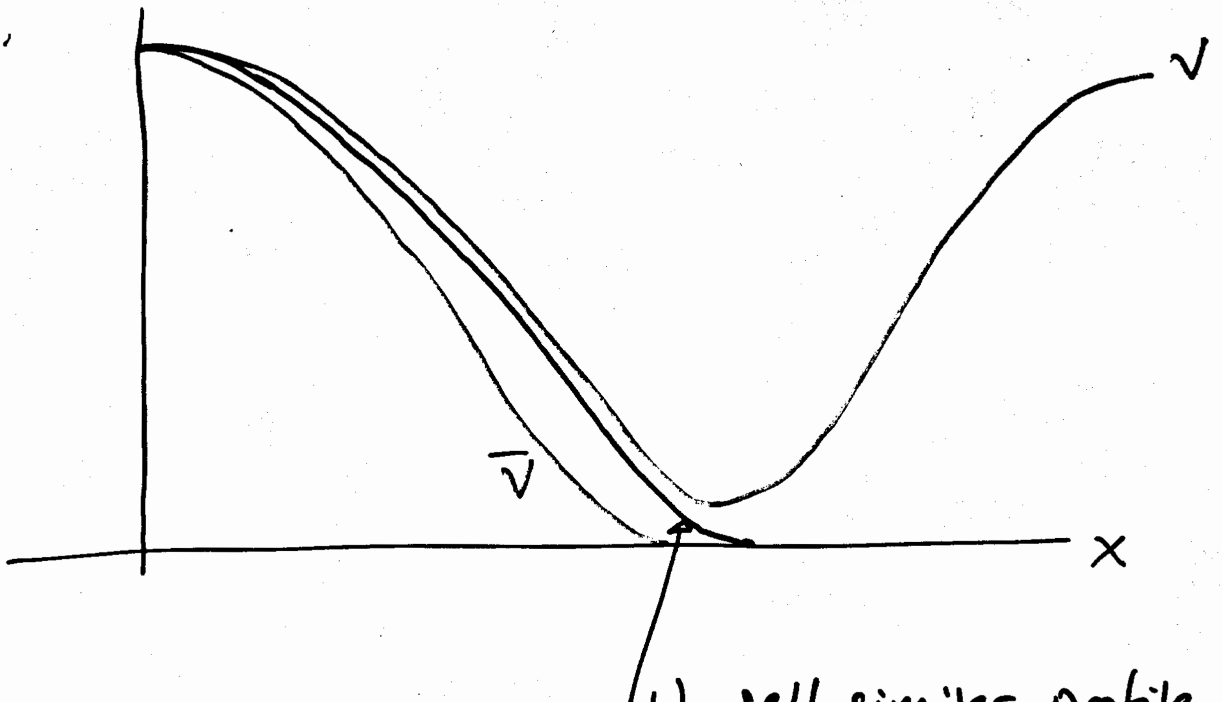
$\bar{V}$ with compact support & zero contact angles

$$\begin{cases} V''' = -V^2 V' \\ V(0) = H \\ V'(0) = 0 \\ V''(0) = -\frac{H^3}{4} \end{cases} \leftarrow \text{important}$$

V positive periodic steady state with $V_{min} \ll 1$

$$\begin{cases} V''' = -V^2 V' \\ V(0) = H \\ V'(0) = 0 \\ V''(0) = -\frac{H^3}{4} + \varepsilon \end{cases}$$

← just a little less concave than $\bar{V}$, V nearly touches down but misses.



U self similar profile with

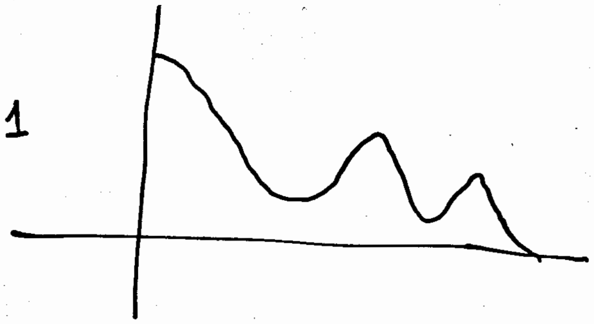
$$\begin{cases} U''' = -\frac{1}{n+4} x U^{1-n} - U^2 U' \\ U(0) = H \\ U'(0) = 0 \\ U''(0) = -\frac{H^3}{4} + \varepsilon \end{cases}$$

if H is sufficiently large then U is trapped between $\bar{V}$ and V with sufficiently tight control to ensure the shooting argument works.

Fix $H>0$.

What is a shooting argument?

case 1

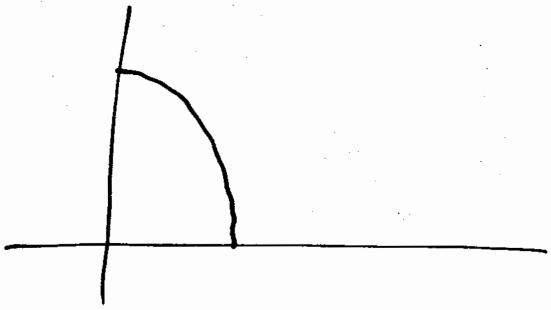


$U(0)=H \quad U'(0)=0$

$U''(0)=\gamma$

γ is not sufficiently negative to ensure only one local max on $[0,L]$

case 2



$U(0)=H \quad U'(0)=0$

$U''(0)=\gamma$

γ is too negative $\Rightarrow U'(L)<0$ at contact line.

$$\alpha_\gamma := \min\{x>0 \mid U'(x)=0 \text{ or } x=\infty\} \leftarrow$$

location of first critical pt, ∞ if $\nexists$

$$\beta_\gamma := \min\{x>0 \mid \lim_{\tilde{x}\uparrow x} U(\tilde{x})=0 \text{ or } x=\infty\}$$

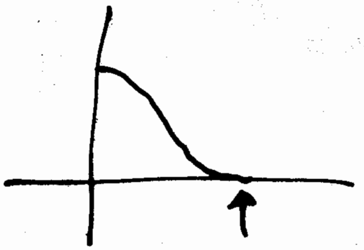
↖ location of contact line or ∞ if $\nexists$

$S^+ := \{\gamma<0 \mid \alpha_\gamma \leq \beta_\gamma\}$ (case 1)

$S^- := \{\gamma<0 \mid \beta_\gamma \leq \alpha_\gamma\}$ (case 2)

If $S^+ \cap S^- \neq \phi$ then

$\gamma \in S^+ \cap S^-$ will yield



zero contact angle.

Standard approach

Assume $S^+ \cap S^- = \phi$.

Show $S^+ \neq \phi$ show $S^- \neq \phi$

show S^+ both open & closed.

Show S^- both open & closed.

$\Rightarrow$ contradiction since

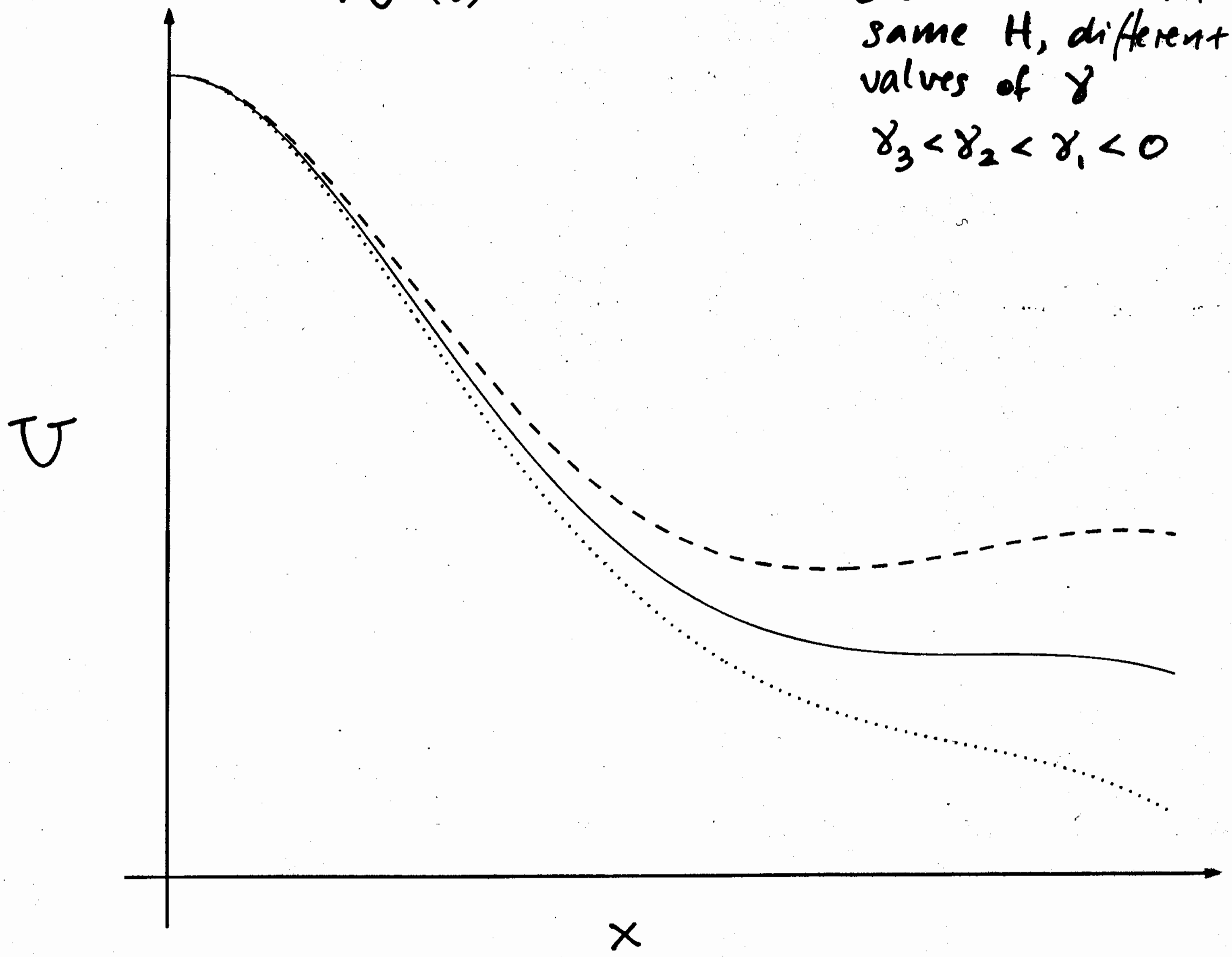
$S^+ \cup S^- = [-\infty, 0]$ ← a connected set.

Problem: In our case, it can be really tricky showing S^+ is open. In fact it's not always true if H isn't sufficiently large.

$$\begin{cases} U''' = -\frac{1}{n+4} x U^{1-n} - U^2 U' \\ U(0) = H \\ U'(0) = 0 \\ U''(0) = \gamma \end{cases}$$

3 solutions with same H, different values of γ

$\gamma_3 < \gamma_2 < \gamma_1 < 0$



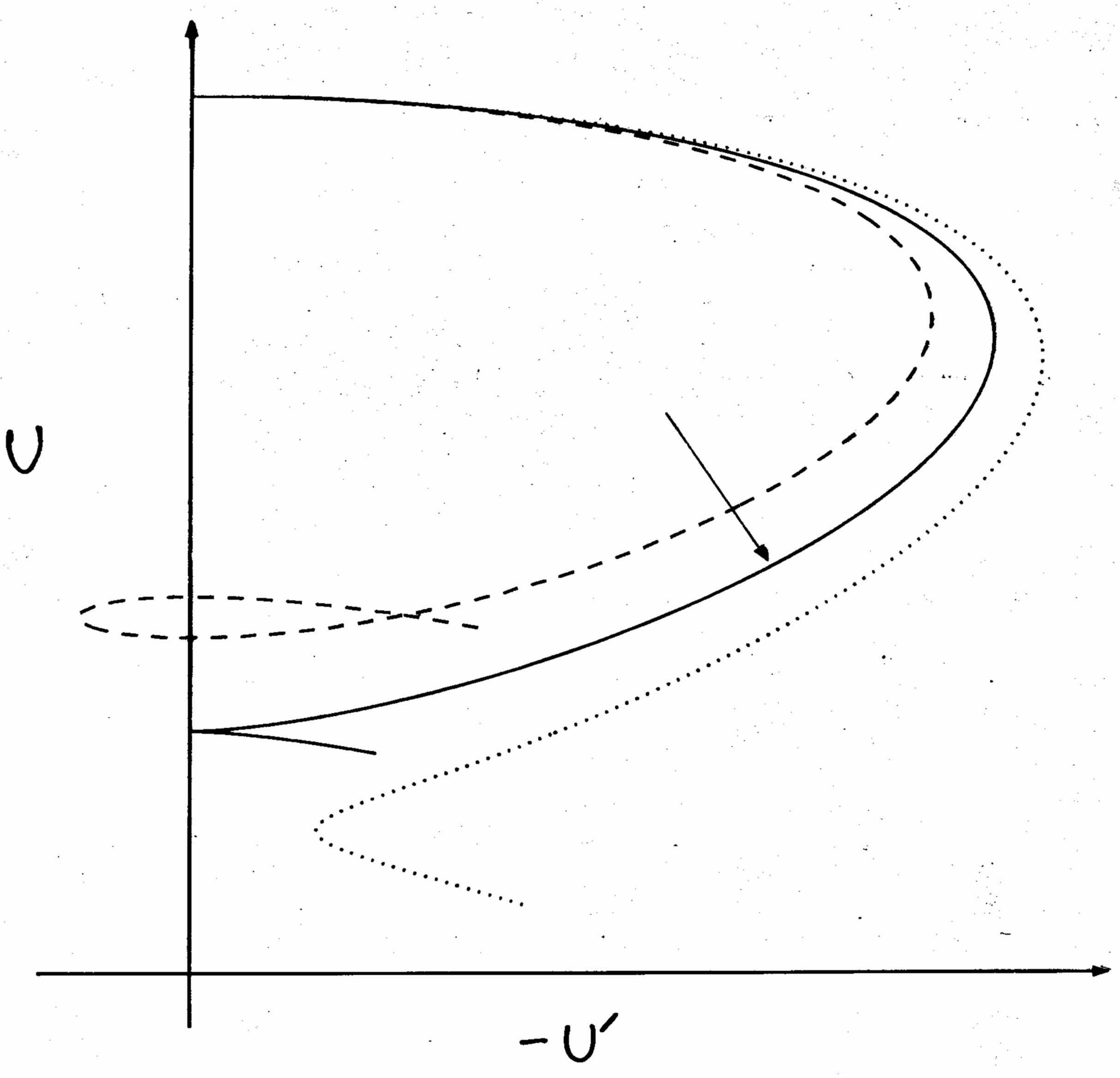
····· : monotonic, $U' < 0$

—— : monotonic, with one critical point $x_0 >$

- - - : not monotonic

$S^+ = \{\gamma < 0 \mid \text{critical point} \leq \text{contact line}\}$

$S^- = \{\gamma < 0 \mid \text{critical point} \geq \text{contact line}\}$



····· : $U''(0) = \gamma_3$

—— : $U''(0) = \gamma_2$

- - - : $U''(0) = \gamma_1$

$\gamma_3 < \gamma_2 < \gamma_1$

$\gamma_2, \gamma_1 \in S^+ \quad \gamma_3 \notin S^+$

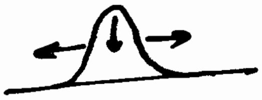
$\Rightarrow S^+$ not open.

Advertisement

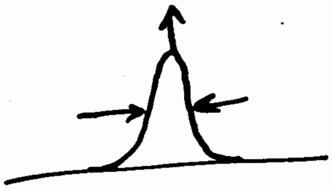
Dejan Slepcev has beautiful new results for the linear stability of

- droplet steady states

- self-similar spreading solutions



- self similar blowup solutions



Nontrivial because

1) equation not 2nd order & linear operator not trivially self adjoint

2) solutions have low regularity at the contact line.