Randomly Modulated Periodic Signals

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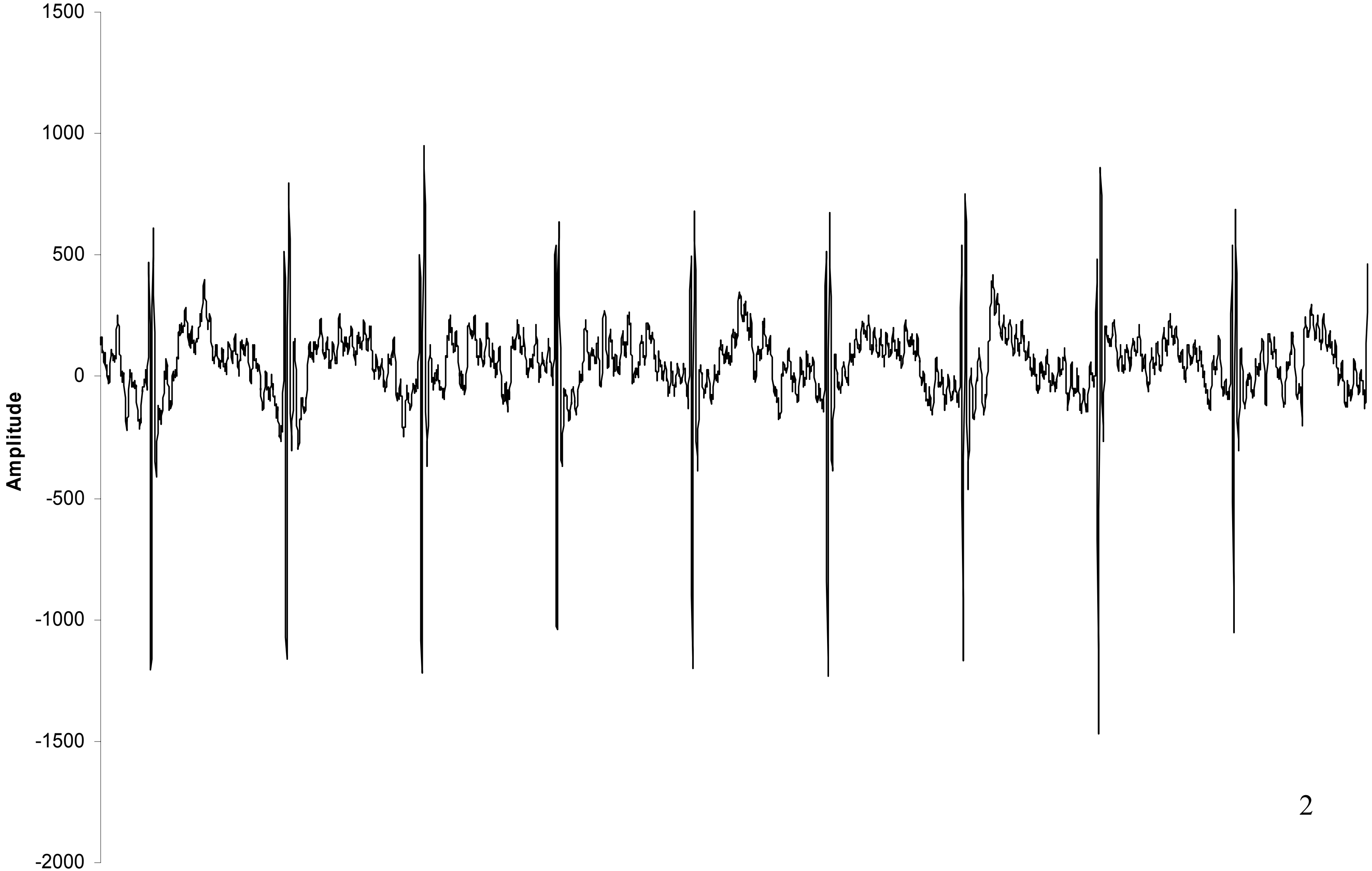
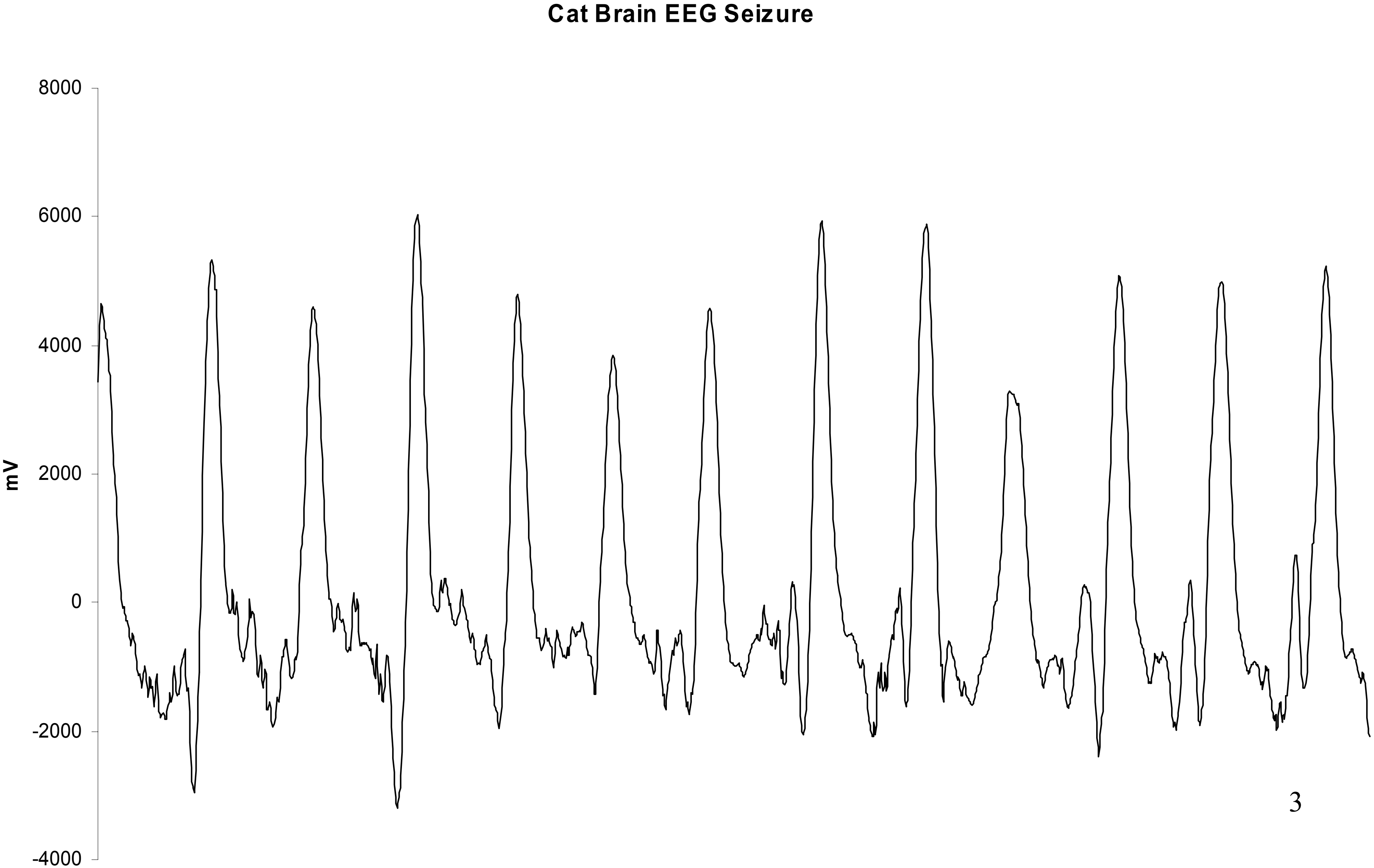
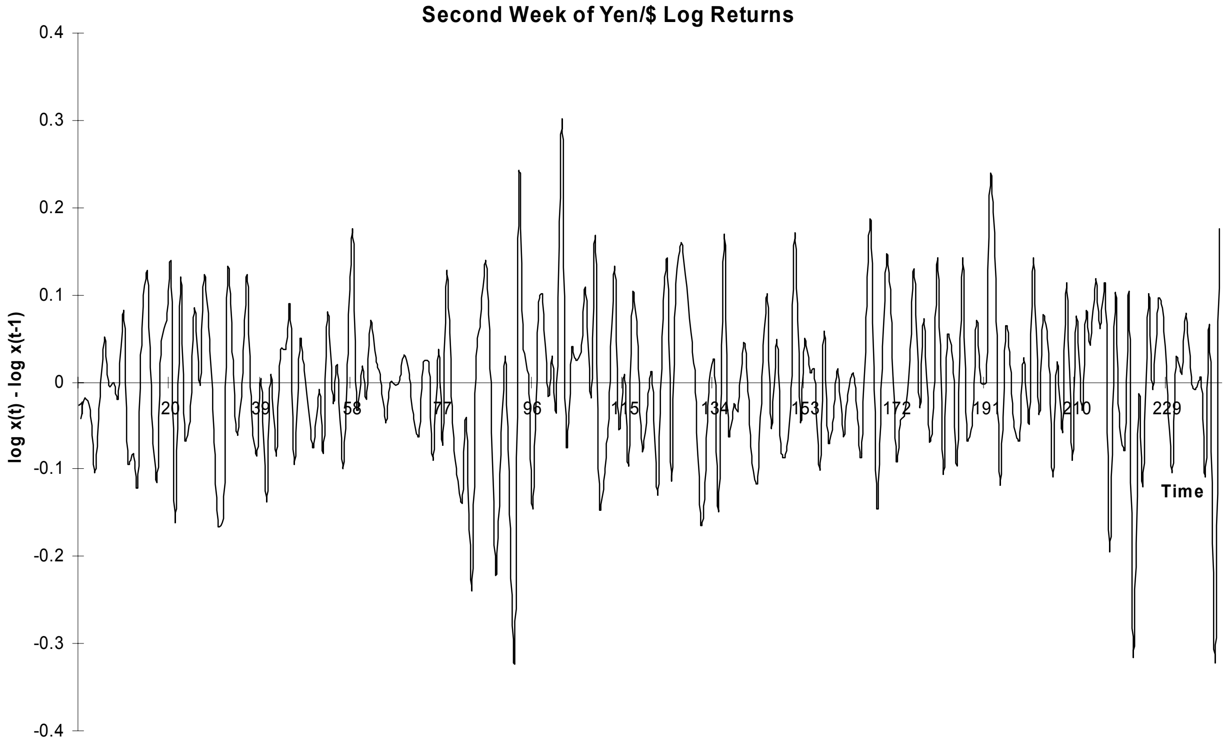
Rotating Cylinder Data



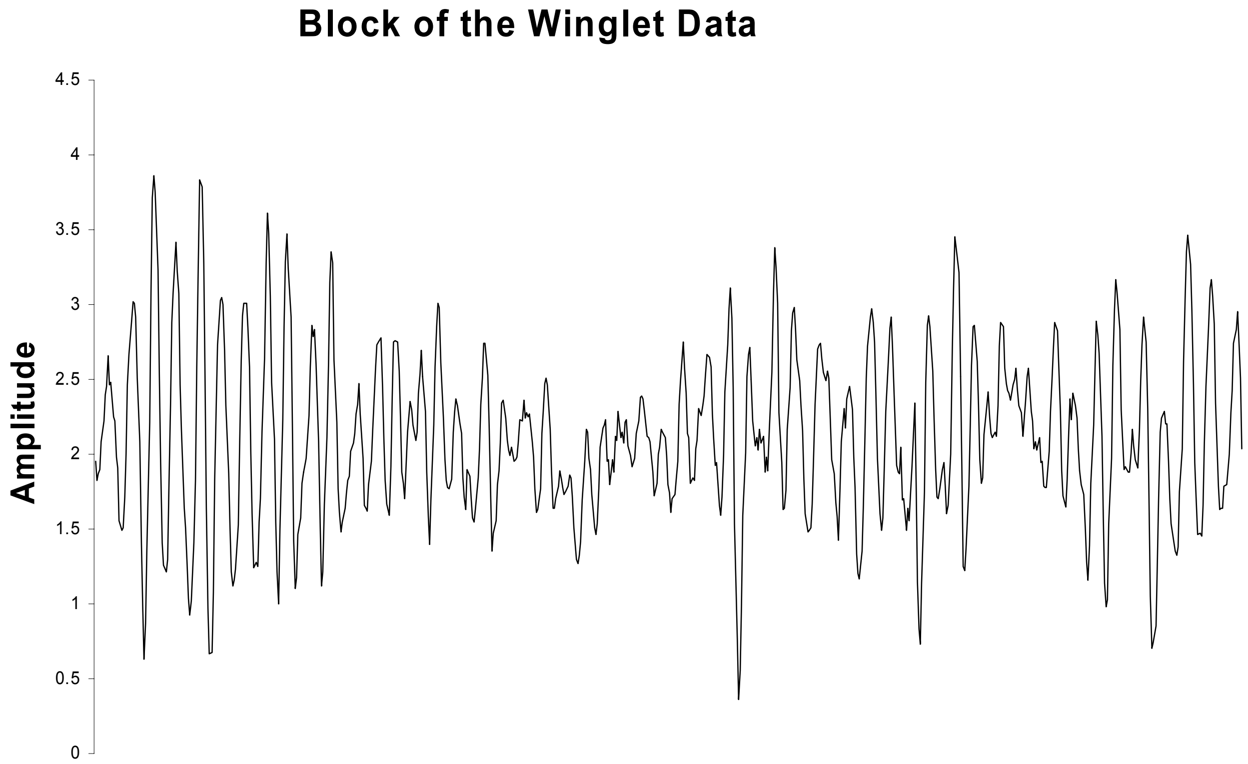
Cat Brain EEG Seizure



One Week of Yen/US$ Returns



Winglet Vibration Data



Definition of a RMP

A signal is called a *randomly modulated periodicity* with period T if it is of the form

$$x(t) = \mu_0 + K^{-1} \sum_{k=1}^{K/2} \begin{bmatrix} \left(s_{1k} + u_{1k}(t)\right)\cos\left(2\pi f_k t\right) + \\ \left(s_{2k} + u_{2k}(t)\right)\sin\left(2\pi f_k t\right) \end{bmatrix}$$

$$f_k = \frac{k}{T} \qquad Eu_{1k}(t) = Eu_{2k}(t) = 0$$

$$\text{for each } k = 1, \cdots, K/2$$

Random Modulations

The vector of the K modulations

$$\mathbf{u}(t)=\{u_{1k}(t),u_{2k}(t):k=1,\cdots,K/2\}$$

are jointly dependent random processes that satisfy two conditions:

Periodic block stationarity

$$f\{\mathbf{u}(t_1),\cdots,\mathbf{u}(t_n)\}=f\{\mathbf{u}(t_1+T),\cdots,\mathbf{u}(t_n+T)\}$$

$$\text{for all } \ 0<t_1<\cdots<t_n<T$$

Finite Dependence

Condition needed to ensure that averaging over frames yields asymptotically gaussian estimates

$$\{\mathbf{u}(t_1),\cdots,\mathbf{u}(t_m)\} \;\;\&\;\; \{\mathbf{u}(t_1'),\cdots,\mathbf{u}(t_n')\}$$

are independently distributed if

$t_m + D < t_1'$ for some D & and all

$$t_1 < \cdots < t_m \quad \& \quad t_1' < \cdots < t_n'$$

Fourier Series for Components

Thus $x(t) = s(t) + u(t)$ where

$$s(t) = s_0 + K^{-1} \sum_{k=1}^{K/2} \left[s_{1k} \cos\left(2\pi f_k t\right) + s_{2k} \sin\left(2\pi f_k t\right) \right]$$

$$u(t) = K^{-1} \sum_{k=1}^{K/2} \left[u_{1k} \cos\left(2\pi f_k t\right) + u_{2k} \sin\left(2\pi f_k t\right) \right]$$

Signal Plus Noise

$s(t)$ is the mean of $x(t)$

$\{u(t)\}$ has a periodic joint distribution

The modulation is part of the signal

It is not measurement noise

Artificial Data Examples

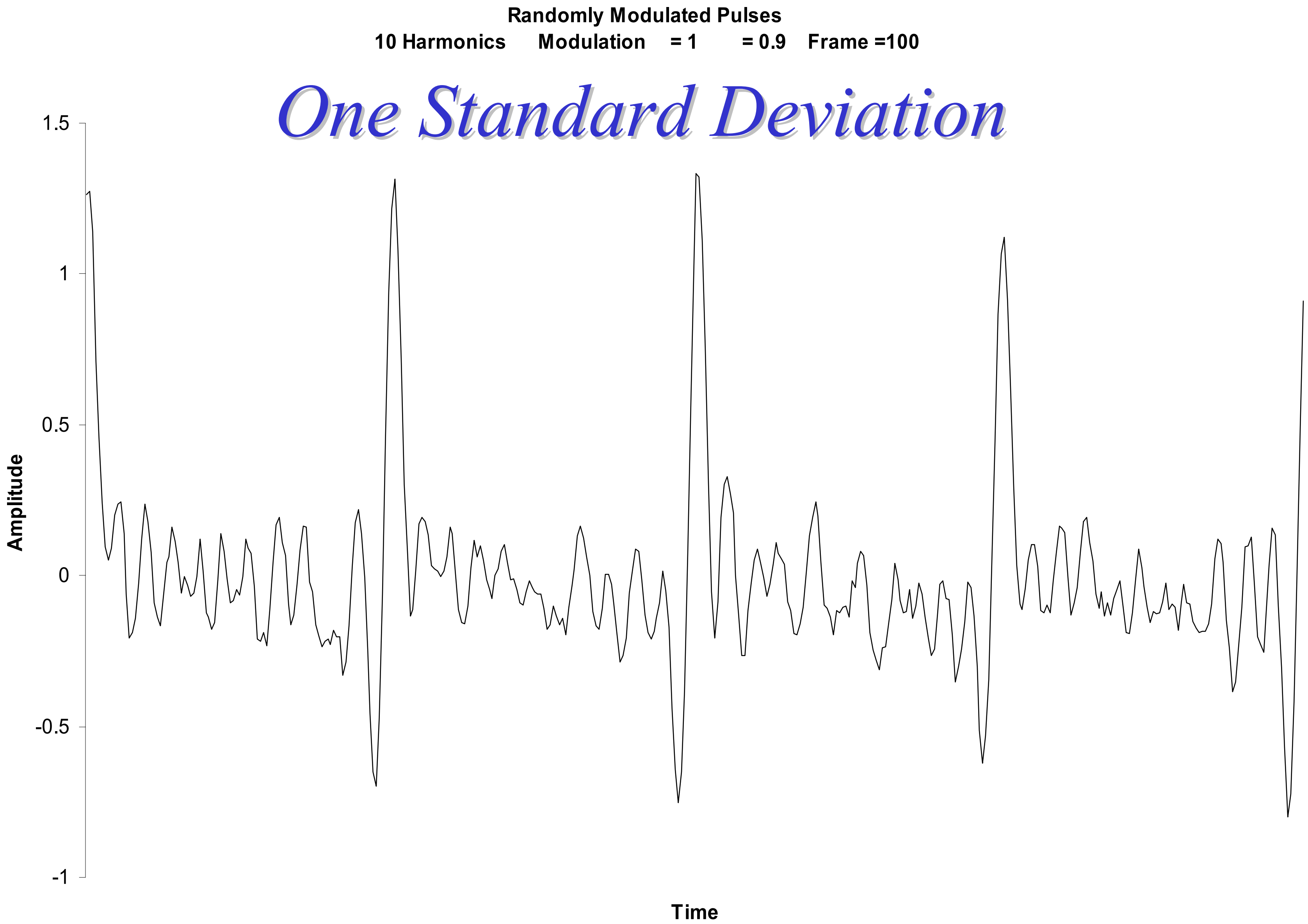
$$x(t) = s_0 + K^{-1} \sum_{k=1}^{K/2} \begin{bmatrix} \left(1+\sigma c_{1k} u_{1k}(t)\right)\cos\left(2\pi f_k t\right) + \\ \left(1+\sigma c_{2k} u_{2k}(t)\right)\sin\left(2\pi f_k t\right) \end{bmatrix}$$

$$u_{1k}(t) = \rho u_{1k}(t-1) + e_1(t)$$

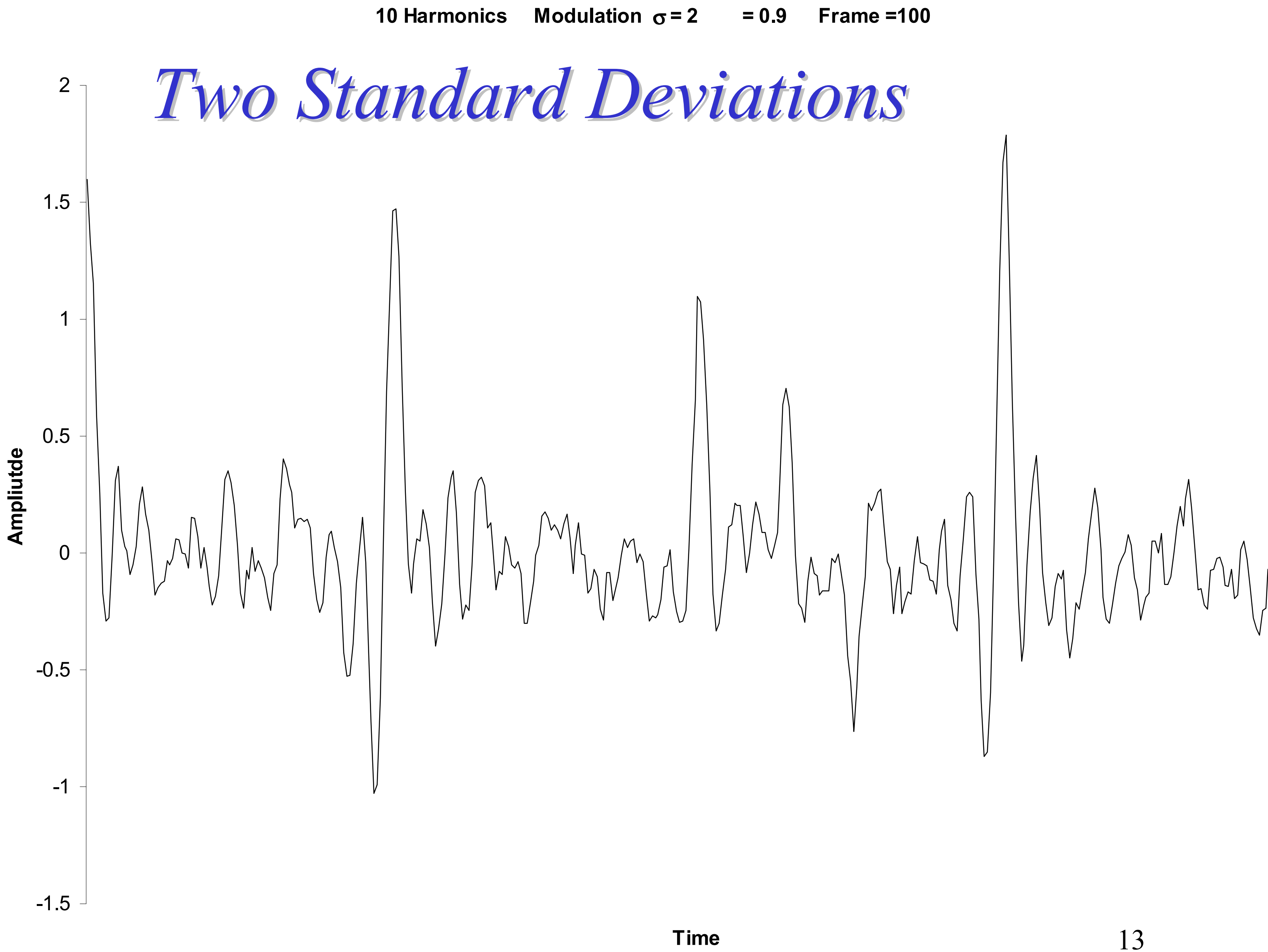
$$u_{2k}(t) = \rho u_{2k}(t-1) + e_2(t)$$

$$c_{jk} = \frac{1}{\max\left[u_{jk}(t) : t = 1, \cdots, N\right]} \qquad j = 1,2$$

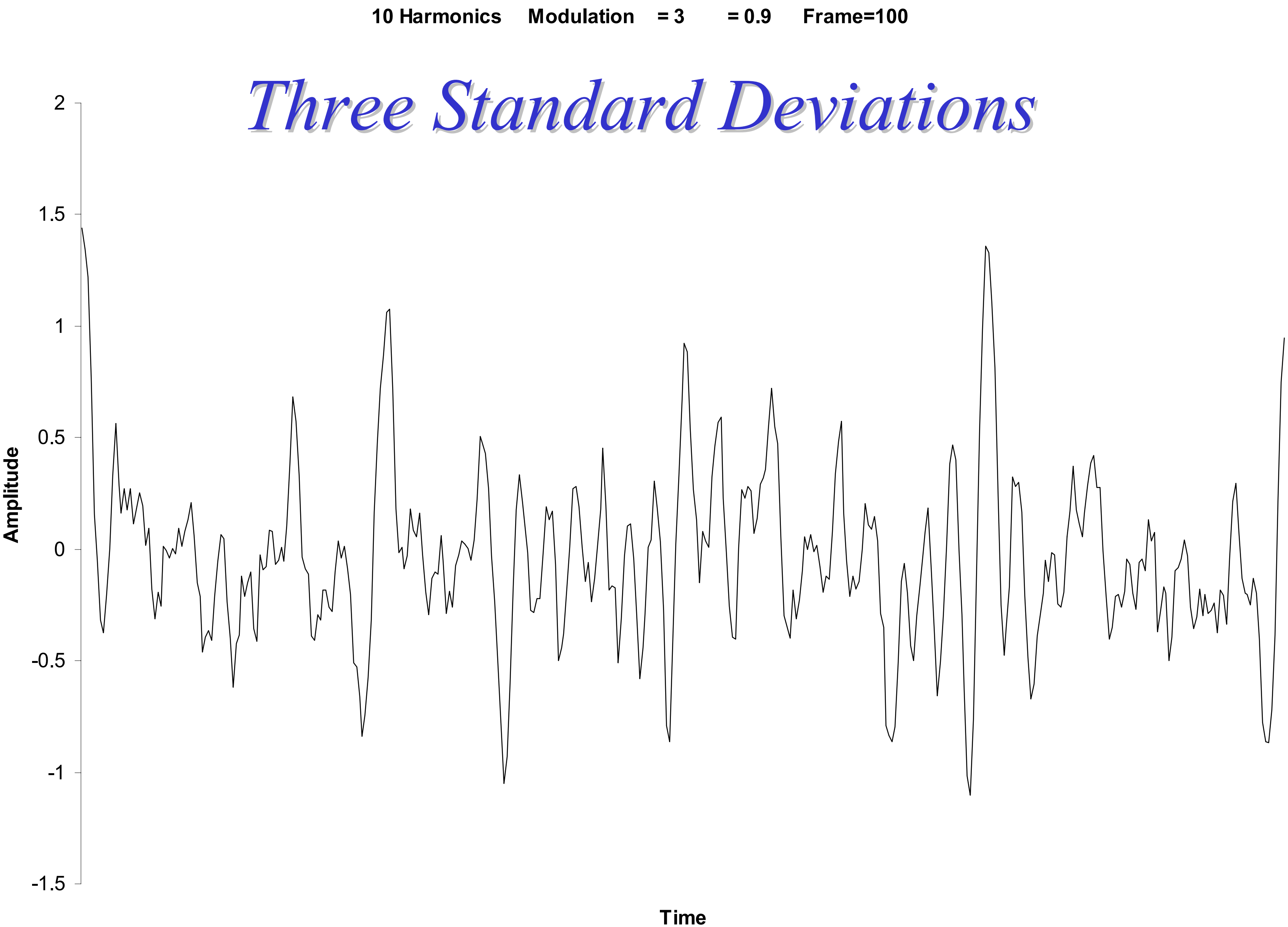
One Standard Deviation



Two Standard Deviations

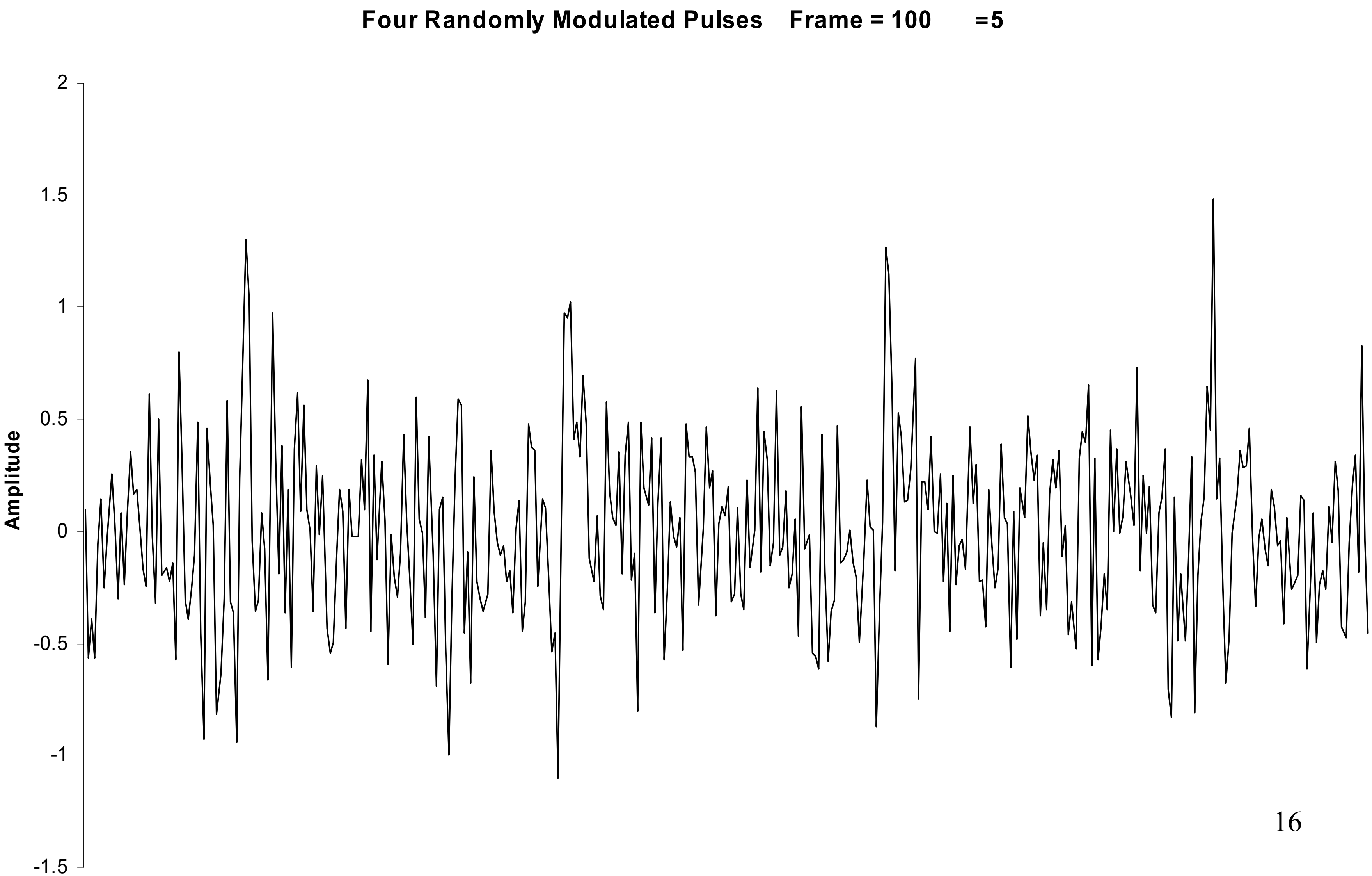


Three Standard Deviations

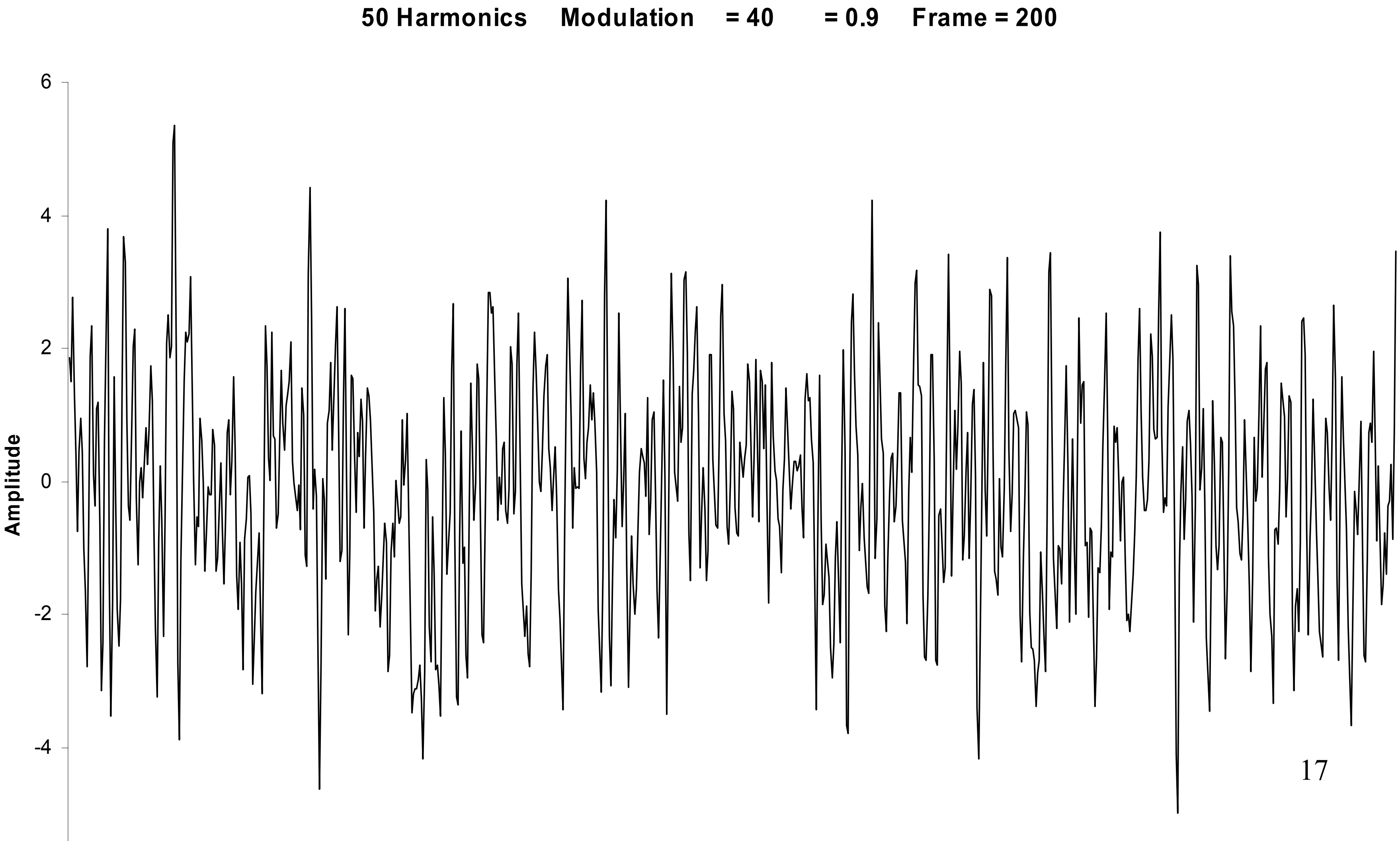


Five Standard Deviations

No Correlation in the Modulation



Noise Like Signal



Block Data into Frames

The data block is divided into M frames of length T

T is chosen by the user to be the period of the periodic component

The *t-th* observation in the *mth* frame is

$$x\big((m-1)T+t\big) \quad t=0,\ldots,T-1$$

Frame Rate Synchronization

The frame length T is chosen by the user to be the hypothetical period of the randomly modulated periodic signal.

If T is not an integer multiple of the true period then coherence is lost.

Signal Coherence Function

$$X_m(k) = \sum_{t=0}^{T-1} x((m-1)T + t)\exp(-i2\pi f_k t)$$

$$X_m(k) = s_k + U_m(k) \qquad s_k = s_{1k} + is_{2k}$$

$$U_m(k) = \sum_{t=0}^{T-1} u_m(t)\exp(-i2\pi f_k t) \ .$$

$$\gamma_x(k) = \sqrt{\frac{|s_k|^2}{|s_k|^2 + E|U_m(k)|^2}}$$

Estimating Signal Coherence

$\{\hat{x}(t) : t = 0, \ldots, T-1\}$ is the mean frame

$$\hat{X}(k) = \sum_{t=0}^{T-1} \hat{x}(t) \exp(-i 2\pi f_k t)$$

$$\hat{\gamma}_x(k) = \sqrt{\frac{\left|\hat{X}(k)\right|^2}{\left|\hat{X}(k)\right|^2 + \hat{\sigma}_u^2(k)}}$$

$$\hat{\sigma}_u^2(k) = M^{-1} \sum_{m=1}^{M} \left| X_m(k) - \hat{X}(k) \right|^2$$

Statistical Measure of Coherence

$$Z(k) = \frac{M}{K\hat{\sigma}_x^2} \left| \hat{X}(k) \right|^2$$

$$\hat{\sigma}_x^2(k) = \left| \hat{X}(k) \right|^2 + \hat{\sigma}_u^2(k)$$

If the modulation is stationary the distribution of each $\mathrm{Z}(\mathrm{k})$ is approximately $\chi_K^2(0)$ & they are indendently distributed.

Coherent Part of the Mean Frame

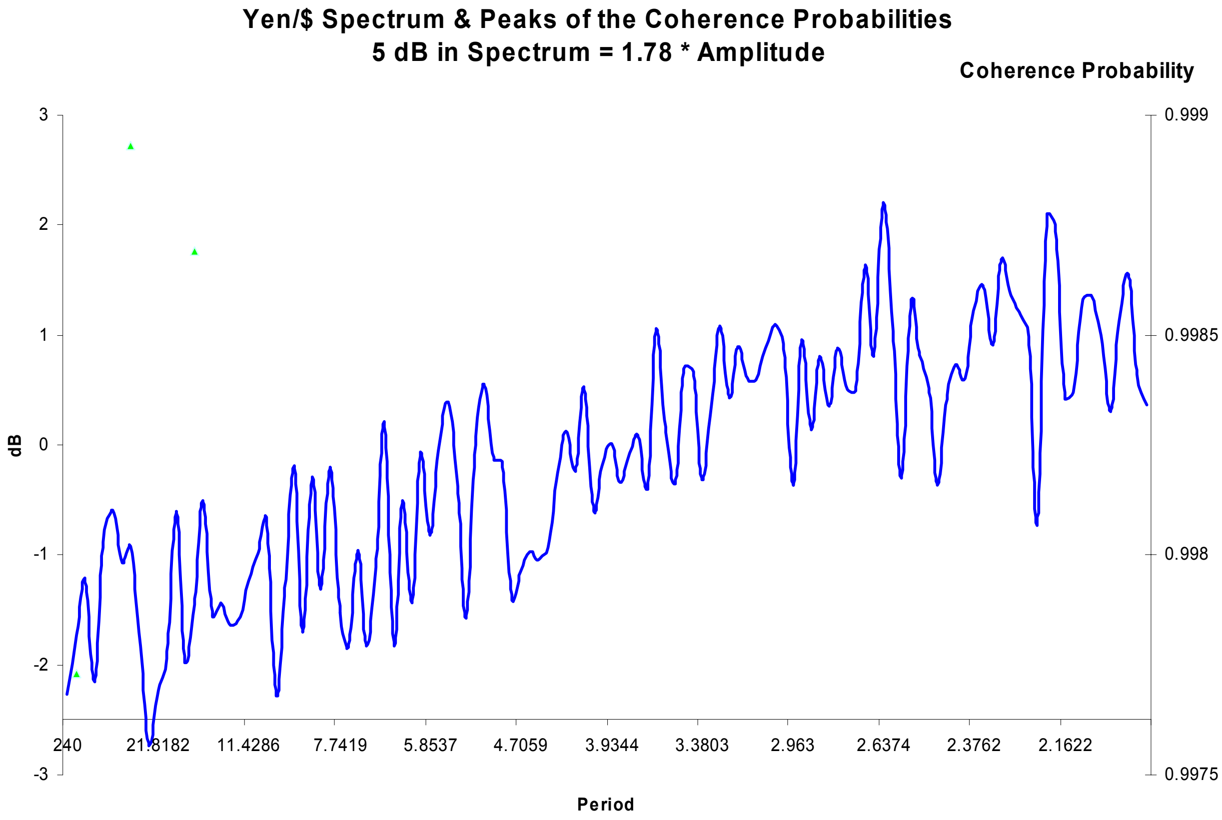
$$\hat{x}_{coherent}(t) = \frac{1}{K}\sum_{k=0}^{K-1} c(k)\,\hat{X}(k)\exp(i2\pi\, f_k t)$$

$$c(k) = 1 \;\text{ if }\; p(k) > threshold \quad c(k) = 0 \; otherwise$$

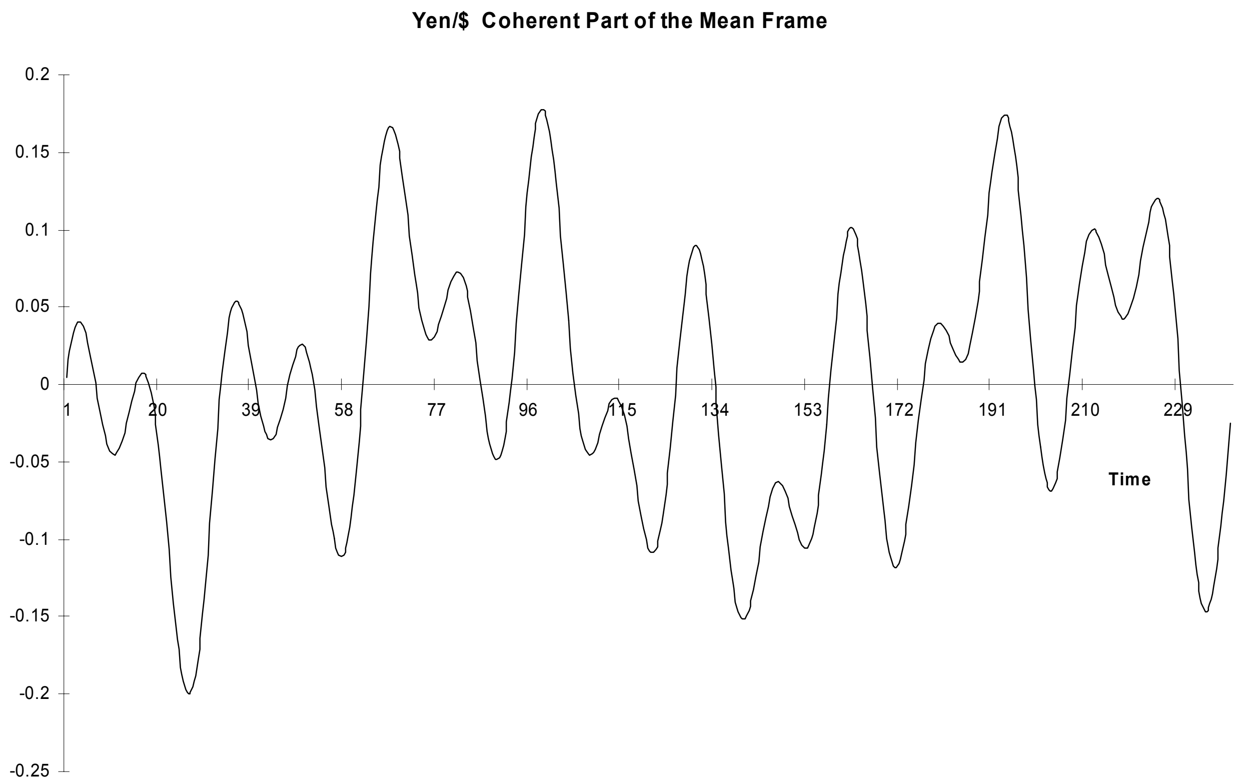
$$p(k) = chisquare\; cdf\left(\chi_2^2\right)$$

$$S(k) = \frac{M}{K\hat{\sigma}_x^2}\left|\hat{X}(k)\right|^2 \quad \sim \quad \chi_2^2$$

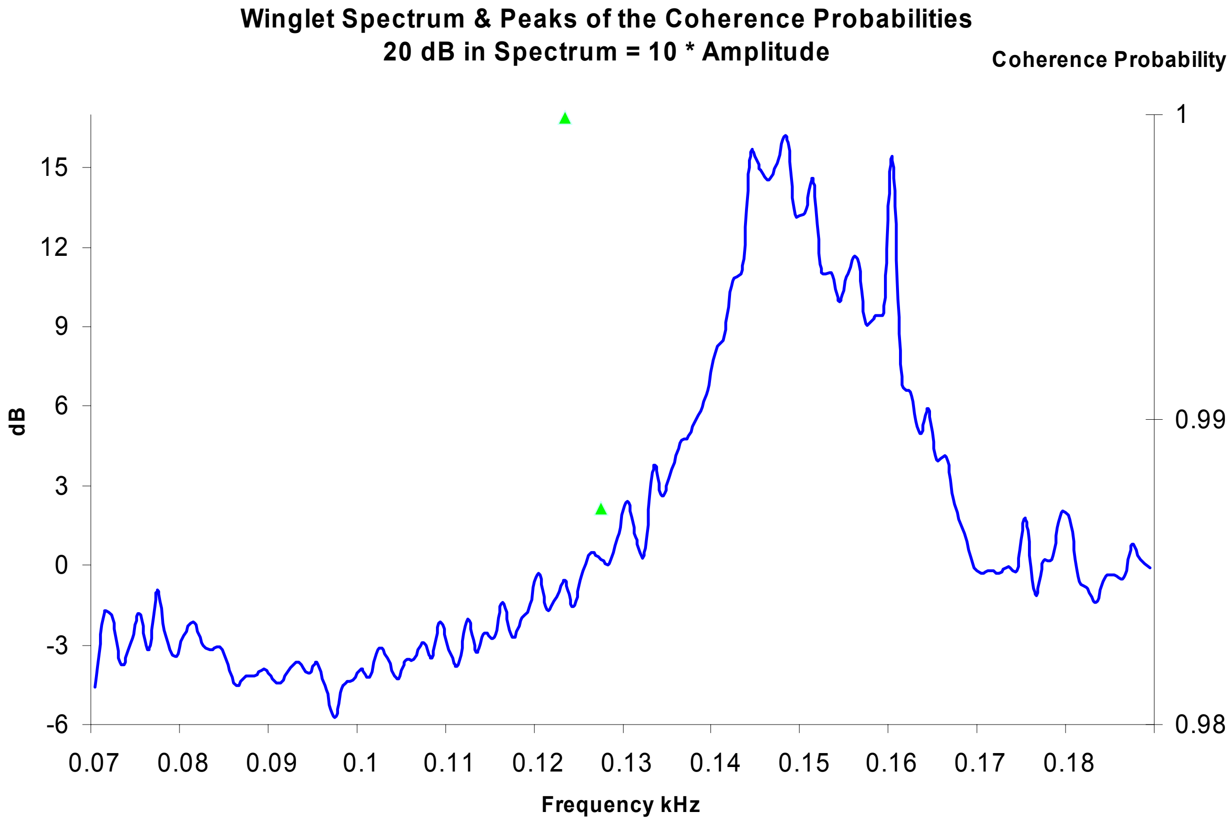
Yen/$ Spectrum & Coherencies



Coherent Part of Yen/$ Mean Frame

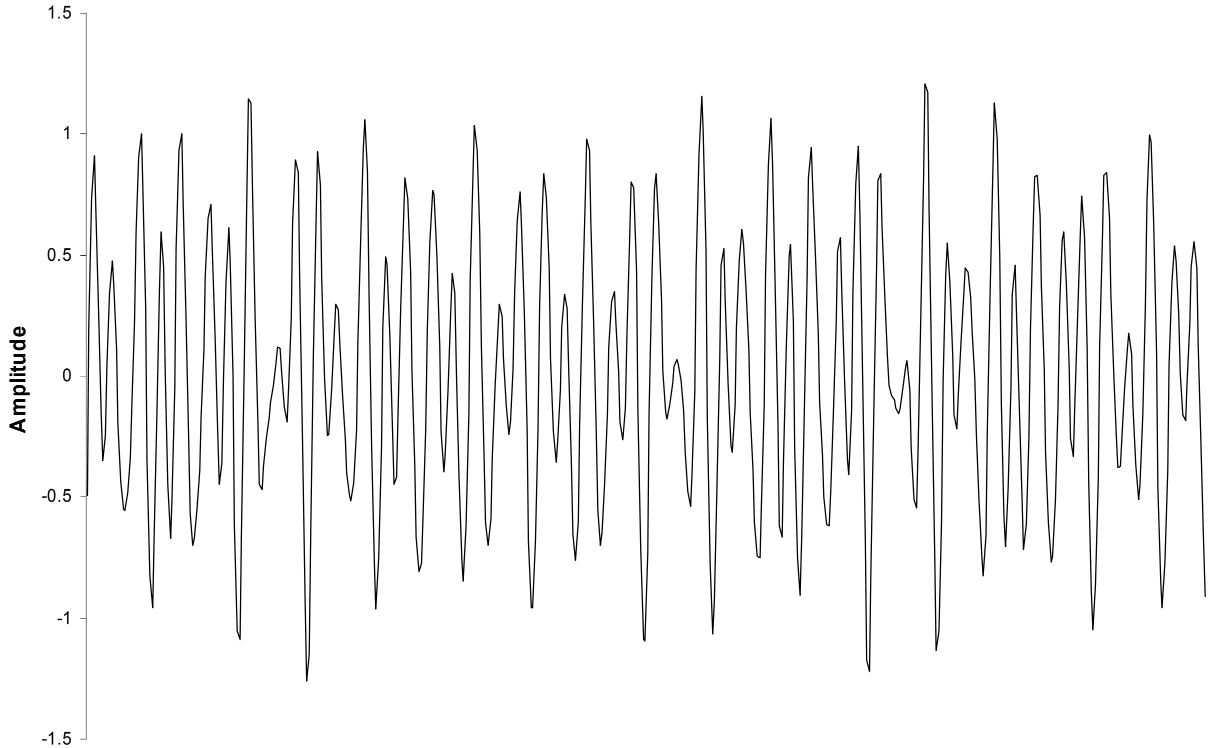


Winglet Spectrum & Coherencies



Coherent Part of Winglet Mean Frame

First Part of the Coherent Part of the Mean Winglet Frame



Exchange Rates Data

25 exchange rate series sampled half-hourly for the whole of 1996 with weekends removed. The weekend period is from 23:00 GMT on Friday when North American financial centres close until 23:00 GMT on Sunday when Australasian markets open. The incorporation of such prices would lead to spurious zero returns and would potentially render trading strategies which recommended a buy or sell at this time to be nonsensical. Removal of these weekend observations leaves 12,575 observations for subsequent analysis.

Summary Statistics

	DEM_ JPY	GBP_ DEM	GBP_ USD	USD_ ITL	USD_ JPY
Mean	3.4E-4	9.7E-4	5.6E-4	-2.1E-4	6.5E-4
Var	6.5E-3	4.6E-3	4.8E-4	9.0E-3	6.2E-3
Skew	-0.049	-0.004	-0.167	-0.011	-0.019
Kurt	5.642	83.51	13.01	15.719	9.723
Min	-0.707	-1.966	-1.137	-0.924	-0.770
Max	0.659	1.992	1.203	0.966	0.758
Lag 1	-0.198	-0.306	-0.205	-0.315	-0.150

Coherence Results I

Period 30 min	Spectrum (dB)	Coherence	*p*-value
	DEM_JPY		
60 hours	-2.164	0.314	0.003
2 hours 2 minutes	-0.165	0.307	0.004
	GBP_DEM		
6 hours	-1.357	0.312	0.003
2 hours 8 minutes	-0.775	0.329	0.002
1 hour 38 minutes	-1.534	0.317	0.003
	GBP_USD		
12 hours	-3.688	0.302	0.005
8 hours	-2.882	0.295	0.007
	USD_CHF		
15 hours	-1.320	0.336	0.001
8 hours	-2.110	0.345	0.001
3 hours 32 minutes	-0.803	0.299	0.006
2 hours	0.096	0.283	0.010

Coherence Results II

	Panel E: USD_DEM		
15 hours	0.390	0.352	0.001
8 hours	-0.365	0.331	0.002
	Panel F: USD_ITL		
12 hours	-4.166	0.308	0.004
5 hours 43 min	-3.135	0.329	0.002
1 hour 49 min	0.139	0.353	0.001
	Panel G: USD_JPY		
60 hours	-1.669	0.341	0.001
15 hours	-0.956	0.362	0.000
8 hours	-1.399	0.349	0.001
2 hours 37 min	0.046	0.298	0.006
1 hour 5 min	0.907	0.297	0.006

30 Minute Exchange Rate Coherencies

One Week Frames