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Query Note

SUBJECT: ***Price Pseudo-Variance, Pseudo-Covariance, Pseudo-Volatility, and Pseudo-Correlation Swaps — In Analytical Close Forms***

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1 Introduction

So-called volatility swaps have become popular in the over-the-counter market. Also, the so-called correlation swaps have become available. It has been pointed out that although participants talk of volatility and correlation swaps, it is in fact variance and covariance swaps that have the more fundamental significance. The complexity of pricing volatility and correlation swaps involves how to determine the dynamics of diffusion processes of underlying rates. However, here we only consider a less-advanced (or simpler) case. Suppose we are only interested in swaps involving the so-called pseudo-statistics, namely the pseudo-variance, -covariance, -volatility, and -correlation. We would like to find analytical close form solutions or approximations.

This brief query note on pricing pseudo-variance, -covariance, -volatility, and -correlation swaps is based on the following papers:

[1] K. Demeterfi, E. Derman, M. Kamal, and J. Zou, "A guide to volatility and variance swaps", *The Journal of Derivatives*, Summer, 1999, pp. 9-32.

[2] O. Brockhaus and D. Long, "Volatility swaps made simple", *Risk*, January, 2000, pp. 92-95.

2 Variance and Covariance Swaps

Let S be a strictly positive underlying rate which follows an Ito process as below

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \ , \quad t > 0 \ , \quad S_0 > 0 \ , \tag{1}$$

or equivalently

$$\mathrm{d} \ln S_t = m_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}W_t \ , \quad t > 0 \ , \tag{2}$$

where

$$m_t = \left(\mu_t - \frac{\sigma_t^2}{2} \right) \ .$$

The basic assumption we impose on the drift term μ and the diffusion term σ is that $\ln S$ given by (2) is a L^2-semi-martingale with the local-martingale part given by the second term on the right side of (2).

2.1 Variance Swaps — Definitions and Assumptions

For any $0 \leq T_{\mathrm{s}} < T_{\mathrm{e}}$, which are respectively called the observation-start time point and the observation-end time point, define

$$\varSigma^2_{(S)}(T_{\mathrm{s}}, T_{\mathrm{e}}) = \frac{1}{T_{\mathrm{e}} - T_{\mathrm{s}}} \int_{T_{\mathrm{s}}}^{T_{\mathrm{e}}} \sigma_\tau^2 \, \mathrm{d}\tau \ , \tag{3}$$

which is called a realized volatility-square over an observation period of $[T_{\mathrm{s}}, T_{\mathrm{e}}]$. It is also easy to see that (3) can be equivalently defined by

$$\varSigma^2_{(S)}(T_{\mathrm{s}}, T_{\mathrm{e}}) = \frac{2}{T_{\mathrm{e}} - T_{\mathrm{s}}} \left(\int_{T_{\mathrm{s}}}^{T_{\mathrm{e}}} \frac{1}{S_\tau} \, \mathrm{d}S_\tau - \ln \frac{S_{T_{\mathrm{e}}}}{S_{T_{\mathrm{s}}}} \right) \ . \tag{4}$$

On the other hand, the concept of the realized volatility-square is closely related to the quadratic variation process of a L^2-semi-martingale. Clearly, we have

$$(T_e - T_s)\Sigma^2_{(S)}(T_s, T_e) = [\ln S]_{T_e} - [\ln S]_{T_s} \ , \tag{5}$$

where $[\cdot]$ is the operator of the quadratic variation. If the local martingale part of $\ln S$ is martingale, then

$$\mathrm{E}_0\left[(T_e - T_s)\Sigma^2_{(S)}(T_s, T_e)\right] = \mathrm{Var}_0\left[\ln S_{T_e}\right] - \mathrm{Var}_0\left[\ln S_{T_s}\right] \ , \tag{6}$$

where $\mathrm{E}_0[\cdot]$ and $\mathrm{Var}_0[\cdot]$ are expectation and variance operators, respectively. Further, if σ is deterministic and $T_s = 0$, then, (6) simply becomes

$$T_e \Sigma^2_{(S)}(0, T_e) = \mathrm{Var}_0\left[\ln S_{T_e}\right] \ . \tag{7}$$

A so-called variance forward contract of the underlying rate S, which is also called a variance swap, is defined as a forward contract with one unit of notional principal, a given maturity of $T > 0$, an observation period of $[T_s, T_e]$ $(0 \le T_s < T_e \le T)$, a strike of Σ^2_K, and a long-short index of I (1 for long and -1 for short) such that the matured payoff of the contract, denoted by $V_{\mathrm{var}}(T)$, is given by

$$V_{\mathrm{var}}(T) = \alpha_{\mathrm{var}} \cdot I \cdot \left[\Sigma^2_{(S)}(T_s, T_e) - \Sigma^2_K\right] \ , \tag{8}$$

where α_{var} is a converting parameter such as \$1 per volatility-square.

Let us first assume that there exists some numeraire pair $(\mathbb{N}, N)$ where $\mathbb{N}$ is the equivalent martingale measure with respect to the numeraire process N. Then the initial value of the contract, denoted by $V_{\mathrm{var}}(0)$, can be given by

$$\begin{aligned} V_{\mathrm{var}}(0) &= \alpha_{\mathrm{var}} \cdot I \cdot \mathrm{E}_0^{\mathbb{N}}\left[\frac{N_0}{N_T}\left(\Sigma^2_{(S)}(T_s, T_e) - \Sigma^2_K\right)\right] \\ &= \alpha_{\mathrm{var}} \cdot I \cdot \mathrm{E}_0^{\mathbb{N}}\left[N_0\left(\Sigma^2_{(S)}(T_s, T_e) - \Sigma^2_K\right)\mathrm{E}_{T_e}^{\mathbb{N}}\left[N_T^{-1}\right]\right] \\ &= \alpha_{\mathrm{var}} \cdot I \cdot \mathrm{E}_0^{\mathbb{N}}\left[\mathrm{df}(T_e, T)\frac{N_0}{N_{T_e}}\left(\Sigma^2_{(S)}(T_s, T_e) - \Sigma^2_K\right)\right] \ , \end{aligned} \tag{9}$$

where $\mathrm{E}_0^{\mathbb{N}}[\cdot]$ is the expectation operator under the measure of $\mathbb{N}$ and $\mathrm{df}(\cdot, \cdot)$ is the zero-coupon bond price.

We then assume that $\mathrm{df}(T_e, T) \cdot N_0 \cdot N_{T_e}^{-1}$ is independent of $\Sigma^2_{(S)}(T_s, T_e)$. Then (9) can be written as

$$V_{\mathrm{var}}(0) = \alpha_{\mathrm{var}} \cdot I \cdot \mathrm{E}_0^{\mathbb{N}}\left[\mathrm{df}(T_e, T)\frac{N_0}{N_{T_e}}\right] \cdot \mathrm{E}_0^{\mathbb{N}}\left[\Sigma^2_{(S)}(T_s, T_e) - \Sigma^2_K\right] \ . \tag{10}$$

If we further assume that the numeraire process N_t can be given by the zero-coupon bond price as

$$N_t = \mathrm{df}(t, T) \ , \qquad 0 \le t \le T \ , \tag{11}$$

then (10) becomes

$$V_{\mathrm{var}}(0) = \alpha_{\mathrm{var}} \cdot I \cdot \mathrm{df}(0, T) \cdot \left(\mathrm{E}_0^{\mathbb{N}}\left[\Sigma^2_{(S)}(T_s, T_e)\right] - \Sigma^2_K\right) \ . \tag{12}$$

Therefore, pricing the variance swap reduces to calculating the expectation of the realized volatility-square.

2.2 Covariance Swaps — Definitions and Assumptions

Now let $S^{(1)}$ and $S^{(2)}$ be two strictly positive Ito's processes given by (1) with $\mu^{(i)}$ and $\sigma^{(i)}$ for $i = 1, 2$, i.e.,

$$\mathrm{d}\ln S_t^{(i)} = m_t^{(i)}\,\mathrm{d}t + \sigma_t^{(i)}\,\mathrm{d}W_t^{(i)}\ , \quad t > 0\ , \tag{13}$$

where

$$m_t^{(i)} = \left(\mu_t^{(i)} - \frac{{\sigma_t^{(i)}}^2}{2}\right)$$

and

$$[\,\mathrm{d}W^{(1)},\,\mathrm{d}W^{(2)}]_t = \rho_t\,\mathrm{d}t\ , \tag{14}$$

in which, $[\cdot,\cdot]$ is the quadratic co-variation operator. Here, $\mu^{(i)}$, $\sigma^{(i)}$ and ρ are such that the basic assumption is valid for $\ln S^{(i)}$, $i = 1, 2$, and

$$\left[\ln S^{(1)}, \ln S^{(2)}\right] = \left[\int \sigma_t^{(1)}\,\mathrm{d}W_t^{(1)}, \int \sigma_t^{(2)}\,\mathrm{d}W_t^{(2)}\right]\ .$$

For any $0 \le T_{\mathrm{s}} < T_{\mathrm{e}}$, define

$$\Sigma^2_{(S^{(1)},S^{(2)})}(T_{\mathrm{s}}, T_{\mathrm{e}}) = \frac{1}{T_{\mathrm{e}} - T_{\mathrm{s}}}\left(\left[\ln S^{(1)}, \ln S^{(2)}\right]_{T_{\mathrm{e}}} - \left[\ln S^{(1)}, \ln S^{(2)}\right]_{T_{\mathrm{s}}}\right)\ . \tag{15}$$

For the time being, let us call it a realized volatility-cross over an observation period of $[T_{\mathrm{s}}, T_{\mathrm{e}}]$. If the local martingale parts of $\ln S^{(1)}$ and $\ln S^{(2)}$ are martingale, then we have

$$\mathrm{E}_0\left[(T_{\mathrm{e}} - T_{\mathrm{s}})\Sigma^2_{(S^{(1)},S^{(2)})}(T_{\mathrm{s}}, T_{\mathrm{e}})\right] = \mathrm{Cov}_0\left[\ln S^{(1)}_{T_{\mathrm{e}}}, \ln S^{(2)}_{T_{\mathrm{e}}}\right] - \mathrm{Cov}_0\left[\ln S^{(1)}_{T_{\mathrm{s}}}, \ln S^{(2)}_{T_{\mathrm{s}}}\right]\ , \tag{16}$$

where $\mathrm{Cov}_0[\cdot,\cdot]$ is the co-variance operators. Further, if $\sigma^{(1)}$, $\sigma^{(2)}$ and ρ are deterministic and $T_{\mathrm{s}} = 0$, then, (16) simply becomes

$$T_{\mathrm{e}}\Sigma^2_{(S^{(1)},S^{(2)})}(0, T_{\mathrm{e}}) = \mathrm{Cov}_0\left[\ln S^{(1)}_{T_{\mathrm{e}}}, \ln S^{(2)}_{T_{\mathrm{e}}}\right]\ . \tag{17}$$

A so-called covariance forward contract of the underlying rates $S^{(1)}$ and $S^{(2)}$, which is also called a covariance swap, is defined as a forward contract with one unit of notional principal, a given maturity of $T > 0$, an observation period of $[T_{\mathrm{s}}, T_{\mathrm{e}}]$ $(0 \le T_{\mathrm{s}} < T_{\mathrm{e}} \le T)$, a strike of Σ^2_{K}, and a long-short index of I such that the matured payoff of the contract, denoted by $V_{\mathrm{cov}}(T)$, is given by

$$V_{\mathrm{cov}}(T) = \alpha_{\mathrm{cov}} \cdot I \cdot \left[\Sigma^2_{(S^{(1)},S^{(2)})}(T_{\mathrm{s}}, T_{\mathrm{e}}) - \Sigma^2_{\mathrm{K}}\right]\ , \tag{18}$$

where α_{cov} is a converting parameter such as \$1 per volatility-cross.

Pricing a covariance swap can be reduced to pricing a variance swap. Let us elaborate. Let us introduce

$$\sigma^2_{\pm}(t) = (\sigma_t^{(1)})^2 \pm 2\rho_t\sigma_t^{(1)}\sigma_t^{(2)} + (\sigma_t^{(2)})^2\ . \tag{19}$$

Applying Levi-Kunita-Watanabe Theorem, one can prove that the local martingale process $W^{(\pm)}$ which is defined by

$$\mathrm{d}W_t^{(\pm)} = \frac{1}{\sigma_{\pm}(t)}\left(\sigma_t^{(1)}\,\mathrm{d}W_t^{(1)} \pm \sigma_t^{(2)}\,\mathrm{d}W_t^{(2)}\right)\ , \qquad W_0^{(\pm)} = 0\ , \tag{20}$$

is a standard Wiener process. Thus we have

$$\begin{aligned}
\mathrm{d}\ln\left(S_t^{(1)}\cdot S_t^{(2)}\right) &= (m^{(1)}+m^{(2)})\,\mathrm{d}t+\sigma_+(t)\,\mathrm{d}W_t^{(+)}\ , &(21)\\
\mathrm{d}\ln\left(S_t^{(1)}/S_t^{(2)}\right) &= (m^{(1)}-m^{(2)})\,\mathrm{d}t+\sigma_-(t)\,\mathrm{d}W_t^{(-)}\ . &(22)
\end{aligned}$$

We also know that

$$\left[\ln S^{(1)},\ln S^{(2)}\right]_t=\frac{1}{4}\left(\left[\ln\left(S^{(1)}\cdot S^{(2)}\right)\right]_t-\left[\ln\left(S^{(1)}/S^{(2)}\right)\right]_t\right)\ .$$

Therefore, we have

$$\Sigma^2_{(S^{(1)},S^{(2)})}(T_\mathrm{s},T_\mathrm{e})=\frac{1}{4}\left(\Sigma^2_{(S^{(1)}\cdot S^{(2)})}(T_\mathrm{s},T_\mathrm{e})-\Sigma^2_{(S^{(1)}/S^{(2)})}(T_\mathrm{s},T_\mathrm{e})\right)\ . \tag{23}$$

With the same argument made in pricing variance swaps, we see that if $N_t=\mathrm{df}(t,T)$ is a numeraire process, then the initial value of the covariance swap, denoted by $V_\mathrm{cov}(0)$, can be given by

$$V_\mathrm{cov}(0)=\frac{1}{4}\alpha_\mathrm{cov}\cdot I\cdot\mathrm{df}(0,T)\cdot\left(\mathrm{E}_0^{\mathbb{N}}\left[\Sigma^2_{(S^{(1)}\cdot S^{(2)})}(T_\mathrm{s},T_\mathrm{e})\right]-\mathrm{E}_0^{\mathbb{N}}\left[\Sigma^2_{(S^{(1)}/S^{(2)})}(T_\mathrm{s},T_\mathrm{e})\right]-4\Sigma_\mathrm{K}^2\right)\ . \tag{24}$$

2.3 The Expectations of Realized Volatility-square and Volatility-cross

From the analysis provided in the pervious sections, pricing a variance or covariance swap is equivalent to calculating the expectation of the realized volatility-square of a rate. It has been proved that the expectation of a realized volatility-square can be evaluated in a diffusion-model-independent way and the variance swap can be replicated by a portfolio of vanilla options, forward contracts and zero-coupond bonds.

3 Pseudo-Variance and Pseudo-Covariance Swaps

The realized volatility-square defined in the above is continuously sampled over a continuum interval $[T_\mathrm{s},T_\mathrm{e}]$. Here we want to consider discretized sampling cases. For given $0\le T_\mathrm{s}<T_\mathrm{e}$ and an integer $n>1$, we are given a finite observation time point set

$$T_\mathrm{s}=t_0<t_1<\cdots<t_n=T_\mathrm{e}\ .$$

For each $i=1,\cdots,n$, we define the log-return for the underlying rate S as

$$X_i=\ln\left(S_{t_i}/S_{t_{i-1}}\right)\ ,\qquad i=1,\cdots,n\ . \tag{25}$$

Define

$$\hat{\Sigma}^2_{(S)}(n;T_\mathrm{s},T_\mathrm{e})=\frac{n}{T_\mathrm{e}-T_\mathrm{s}}\left(\frac{1}{n-1}\sum_{i=1}^n(X_i-\bar{X}_n)^2\right)\ ,\qquad \bar{X}_n=\frac{1}{n}\sum_{i=1}^n X_i\ , \tag{26}$$

which may be called the realized pseudo-volatility-square, and

$$\begin{aligned}
\hat{\Sigma}^2_{(S^{(1)},S^{(2)})}(n;T_\mathrm{s},T_\mathrm{e}) &= \frac{n}{T_\mathrm{e}-T_\mathrm{s}}\left(\frac{1}{n-1}\sum_{i=1}^n\left(X_i^{(1)}-\bar{X}_n^{(1)}\right)\left(X_i^{(2)}-\bar{X}_n^{(2)}\right)\right)\\
&= \frac{1}{4}\left(\hat{\Sigma}^2_{(S^{(1)}\cdot S^{(2)})}(n;T_\mathrm{s},T_\mathrm{e})-\hat{\Sigma}^2_{(S^{(1)}/S^{(2)})}(n;T_\mathrm{s},T_\mathrm{e})\right)\ , &(27)
\end{aligned}$$

which may be called the realized pseudo-volatility-cross. In the similar way, we may define variance and covariance swaps which are respectively related to the realized pseudo-volatility-square and pseudo-volatility-cross.

Without the loss of generality, it suffices to consider variance swaps. Strictly speaking, a variance swap is the derivative whose underlying driving force is given by the difussion process of a given rate. Let us consider a less-advanced case. Suppose that the underlying rate process S is given by (1) in which μ and σ are deterministic functions of t and S_t. In this case, we call a variance swap of the realized pseudo-volatility-square given by (26) a pseudo-variance swap. In a similar way, we may define a pseudo-covariance swap. Actually, a pseudo-variance swap of the underlying rate S is a forward contract with non-linear path-dependent payoff related to S.

3.1 Price Pseudo-Variance and Pseudo-Covariance Swaps

Under the same pricing framework mentioned before, pricing a pseudo-variance swap is also equivalent to evaluating the expectation of the realized pseudo-volatility-square. Since the model of the underlying rate is given, we can always use Monte-Carlo simulation approach to get the approximate value of the expectation. However, that is not the way we would like to use. Hence, here is our first question.

Question 1

Let μ and σ be deterministic functions of t only. For any given $n > 1$, what is the close form solution or approximation of

$$\mathrm{E}_0^{\mathrm{IN}}\left[\hat{\Sigma}^2_{(S)}(n; T_\mathrm{s}, T_\mathrm{e})\right] ?$$

3.2 Price Pseudo-Volatility and Pseudo-Correlation Swaps

Pseudo-volatility and pseudo-correlation swaps can be considered as the secondary products of the pseudo-variance swaps. Let us define

$$\hat{\sigma}_{(S)}(n; T_\mathrm{s}, T_\mathrm{e}) = \sqrt{\hat{\Sigma}^2_{(S)}(n; T_\mathrm{s}, T_\mathrm{e})} \,, \tag{28}$$

which can be called the realized pseudo-volatility. Let us introduce the following notations.

$$A_1(n) = \hat{\Sigma}^2_{(S^{(1)})}(n; T_\mathrm{s}, T_\mathrm{e}) \,, \tag{29}$$

$$A_2(n) = \hat{\Sigma}^2_{(S^{(2)})}(n; T_\mathrm{s}, T_\mathrm{e}) \,, \tag{30}$$

$$A_{12}(n) = \hat{\Sigma}^2_{(S^{(1)} \cdot S^{(2)})}(n; T_\mathrm{s}, T_\mathrm{e}) \,, \tag{31}$$

$$A_{1/2}(n) = \hat{\Sigma}^2_{(S^{(1)}/S^{(2)})}(n; T_\mathrm{s}, T_\mathrm{e}) \,. \tag{32}$$

Then we define

$$\hat{\rho}_{(S^{(1)}, S^{(2)})}(n; T_\mathrm{s}, T_\mathrm{e}) = \frac{1}{4} \frac{A_{12}(n) - A_{1/2}(n)}{\sqrt{A_1(n)}\sqrt{A_2(n)}} \,, \tag{33}$$

which can be called the realized pseudo-correlation.

A so-called pseudo-volatility forward contract of the underlying rate S, which is also called a pseudo-volatility swap, is defined as a forward contract with one unit of notional principal, a given maturity of $T > 0$, $n+1$ observation times of $T_s = t_0 < t_1 < \cdots < t_n = T_e \leq T$, a strike of σ_K, and a long-short index of I such that the matured payoff of the contract, denoted by $V_{\text{vol}}(T)$, is given by

$$V_{\text{vol}}(T) = \alpha_{\text{vol}} \cdot I \cdot \left[\hat{\sigma}_{(S)}(T_s, T_e) - \sigma_K\right] \ , \tag{34}$$

where α_{vol} is a converting parameter such as \$1 per volatility. Similarly, a so-called pseudo-correlation forward contract of two underlying rates $S^{(1)}$ and $S^{(2)}$, which is also called a pseudo-correlation swap, is defined as a forward contract with one unit of notional principal, a given maturity of $T > 0$, $n+1$ observation times of $T_s = t_0 < t_1 < \cdots < t_n = T_e \leq T$, a strike of ρ_K, and a long-short index of I such that the matured payoff of the contract, denoted by $V_{\text{corr}}(T)$, is given by

$$V_{\text{corr}}(T) = \alpha_{\text{corr}} \cdot I \cdot \left[\hat{\rho}_{(S^{(1)}, S^{(2)})}(T_s, T_e) - \rho_K\right] \ , \tag{35}$$

where α_{corr} is a converting parameter such as \$1 per correlation.

With the same arguments, pricing a pseudo-volatility swap and pseudo-correlation swap are respectively equivalent to evaluating the expectations of the realized pseudo-volatility and the realized pseudo-correlation. The pseudo-volatility $\hat{\sigma}_{(S)}(n; T_s, T_e)$ is the non-linear function of $\hat{\Sigma}^2_{(S)}(n; T_s, T_e)$ (see (28)), and the pseudo-correlation $\hat{\rho}_{(S^{(1)}, S^{(2)})}(n; T_s, T_e)$ is the non-linear function of A_1, A_2, A_{12}, and $A_{1/2}$ (see (33)). To get second-order approximations of the expectations by using Taylor expansion of the functions, we need to calculate some variances and covariances. Thus, we have the following two questions.

Question 2 Let μ and σ be deterministic functions of t only. For any given $n > 1$, what is the close form solution or approximation of

$$\text{Var}_0^{\mathbb{N}}\left[\hat{\Sigma}^2_{(S)}(n; T_s, T_e)\right] \ ?$$

Question 3 Let $\mu^{(i)}$, $\sigma^{(i)}$ and ρ be deterministic functions of t only, $i = 1, 2$. For any given $n > 1$, what are the close form solutions or approximations of

$$\begin{aligned}
&\text{Cov}_0^{\mathbb{N}}\left[A_1(n), A_2(n)\right] \ ? \\
&\text{Cov}_0^{\mathbb{N}}\left[A_1(n), A_{12}(n)\right] \ ? \\
&\text{Cov}_0^{\mathbb{N}}\left[A_1(n), A_{1/2}(n)\right] \ ? \\
&\text{Cov}_0^{\mathbb{N}}\left[A_2(n), A_{12}(n)\right] \ ? \\
&\text{Cov}_0^{\mathbb{N}}\left[A_2(n), A_{1/2}(n)\right] \ ? \\
&\text{Cov}_0^{\mathbb{N}}\left[A_{12}(n), A_{1/2}(n)\right] \ ?
\end{aligned}$$