Risk-neutral Probablility Measure

Miro Powojowski

A fundamental concept in math finance is that of risk-neutral probability measure $F(s)$ for a traded security S. The task of pricing a derivative instrument is equivalent to computing the expectation of the payoff under the risk-neutral measure in the case of European-style options and to maximising that expectation over all stopping times in the case of American-style options.

The most widely-used model for equity stock prices is the geometric Brownian motion, resulting in the celebrated Black-Sholes formula for the price of a European call. One consequence of this model is the lognormal risk-neutral probability measure. However, it has long been recognised that the observed risk-neutral measures are not lognormal, as exhibited by the so called "volatility smile."

Estimation of the risk-neutral measure (sometimes referred to as implied probabilities in the literature) without the assumption of lognormality has been attempted by some researchers (e.g. Rubinstein), along with qualitative comparisons of the estimated measures. However, no inference methods have been proposed to allow statistical testing and comparisons.

The proposed problem deals with the estimation of the risk-neutral measure $F(s)$ from the observed option prices. In order to use the estimated measure $\hat{F}(s)$ in analysing market conditions, two questions must be answered. Firstly, some characteristics of the estimated measure $\hat{F}(s)$ have to be identified as useful market indicators. Usually, these characterisics can be expressed as some functionals μ of $\hat{F}(s)$ and are therefore random variables. One example is the standard deviation of the random variable $log(S)$ under $\hat{F}(s)$ (which can be expressed as $\mu(\hat{F}) = (\int_0^\infty log(s) d\hat{F}(s))^{1/2}$), commonly referred to as the implied volatility. Another example would be $\mu(\hat{F}) = \hat{F}(s_0)$, for any fixed s_0, which is the price of a binary option on S with strike s_0. Secondly, statistical properties of $\hat{F}(s)$ must be examined in order to asses the dispersion of the random variables $\mu(\hat{F})$. Since each criterion $\mu(\hat{F})$ is a statistic of the observations it would be useful to develop tests for statistical hypothesis concerning the estimated criteria. For example, it would be useful to identify statistically significant differences in estimates based on different datasets in order to identify changing market conditions. Such tests could be of tangible interest from the risk-management point of view.

It is hoped that some approaches to these statistical inference questions can be developed during the workshop. The related issues of constructing alternative (to those suggested in the literature) estimators $\hat{F}(s)$ and useful criteria $\mu(\hat{F})$ can also be explored if there is sufficient interest among the participants.

Some market data will be available for testing hypotheses and proposed approaches.