June 8, 2001

Monte Carlo Simulation in the Integrated Market and Credit Risk Portfolio Model

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The Problem

Estimation of risk of the large portfolios of credit risky securities is the problem that can be studied using Monte Carlo methods. The main difficulties include the large number of risk factors (interest rates, fx rates, ...) and statistical dependencies between probabilities of default and market risk factors. There are several variance reduction techniques (importance sampling, stratifies sampling, ...) that are applicable to many practical problems in finance, in particular, in pricing of sophisticated securities. The problem is how to use these techniques for portfolio risk analysis. The most interesting practical case corresponds to credit risky portfolios. In this case the portfolio losses depend on default events that are relatively rear. Therefore, efficient Monte Carlo simulation could be based on a transformation of the measure that describes joint evolution of market and credit risk factors.

The problem can be considered in both one step and multi step setting.

A framework for credit risk estimation that has been used in the industry is based on a joint market credit risk model, described in Iscoe, Kreinin and Rosen 1999. Some of the mathematical details are described below.

The Model

A major limitation of all current Portfolio Credit Risk (PCR) models is the assumption that market risk factors, such as interest rates and foreign exchange rates, are deterministic. Hence, they do not account for stochastic exposures. While this assumption has less consequence for portfolios of loans or floating rate instruments, it has great impact on derivatives such as swaps and options. Ultimately, a comprehensive framework requires the full integration of market and credit risk.

The main idea behind the framework is that conditional on a market scenario all defaults and rating changes are independent. A state-of-the-world is a complete specification at a point in time of the relevant economic and financial credit drivers and market factors (macroeconomic, microeconomic, financial, industrial, etc.) that drive the model. The framework consists of five parts:

Part 1. Risk factors and scenarios

Consider the single period $[t_0, t]$ where, generally, t = 1 year. In this single period model a scenario corresponds to a state-of-the-world. At the end of the horizon, t, the scenario is defined by q^c systemic factors, the credit drivers, which influence the credit worthiness of the obligors in the portfolio.

Denote by $\boldsymbol{x}(t)$ the vector of factor returns at time t; i.e., $\boldsymbol{x}(t)$ has components $x_k(t) = ln\{r_k(t)/r_k(t_0)\}$, where $r_k(t)$ is the value of the k-th factor at time t. Assume that at the horizon the returns are normally distributed: $\boldsymbol{x}(t) \sim N(\mu, \boldsymbol{Q})$, where μ is a vector of mean returns and $\boldsymbol{Q}$ is a covariance matrix. Denote by $\boldsymbol{Z}(t)$, the vector of normalized factor returns; i.e., $Z_k(t) = (\boldsymbol{x}_k(t) - \mu_k) / \sigma_k$.

Part 2. Joint default model

The joint default model consists of three components. First, the definition of unconditional default probabilities. Second, the definition of a credit worthiness index for each obligor and the estimation of a multi-factor model that links the index to the credit drivers. Finally, a model of obligor default, which links the credit worthiness index to the probabilities of default, is used to obtain conditional default probabilities. Below, we explain these components in more detail.

Denote by τ_j the time of default of obligor j, and by $p_j(t)$ its **unconditional probability of default,** the probability of default of an obligor in sector j by time t:

$$p_j(t) = Pr\{\tau_j \le t\} \tag{1}$$

Note that all obligors in sector j have the same unconditional probability of default. We assume that unconditional probabilities for each sector are available from an internal model or from an external agency.

The **credit worthiness index**, Y_j, of obligor j determines the credit worthiness or financial health of that obligor at time t. Whether an obligor is in default can be determined by considering the value of its index. We assume that Y_j, a continuous variable, is related to the credit drivers through a linear, multi-factor model:

$$Y_j(t) = \sum_{k=1}^{q^c} \beta_{jk} Z_k(t) + \sigma_j \varepsilon_j \tag{2}$$

where

$$\sigma_j = \sqrt{1 - \sum_{k=1}^{q^c} \beta_{jk}^2}$$

is the volatility of the idiosyncratic component associated with sector j, β_{jk} is the sensitivity of the index of obligor j to the k-th factor and ε_j, $j = 1, 2, \ldots, N$, are independent and identically distributed standard normal variables. Thus, the first term on the right side of Equation (2) is the systemic component of the index while the second term is the specific, or idiosyncratic, component. Note that the distribution of the index is standard normal; it has zero mean and unit variance.

Since all obligors in a sector are statistically identical, obligors in a given sector share the same multi-factor model. However, while all obligors in a sector share the same β_{jk} and σ_j, each has its own idiosyncratic, uncorrelated component, ε_j.

The **conditional probability of default** of an obligor in sector j, $p_j(t; \mathbf{Z})$, is the probability that an obligor in sector j defaults at time t, conditional on scenario $\mathbf{Z}$:

$$p_j(t;\mathbf{Z}) = Pr\{\tau_j \le t | \mathbf{Z}(t)\} \tag{3}$$

The estimation of conditional probabilities requires a conditional default model which describes the functional relationship between the credit worthiness index Y_j (and hence the systemic factors) and the default probabilities p_j.

We assume that default is driven by a Merton model (Merton 1974). In the Merton model default occurs when the assets of the firm fall below a given boundary or threshold, generally given by its liabilities. We consider that an obligor defaults when its credit worthiness index, Y_j, falls below a pre-specified threshold estimated from historical data. In this setting, an obligor's credit worthiness index, Y_j, can be interpreted as the standardized return of its asset levels. Default occurs when this index falls below α_j, the **unconditional default threshold.**

The unconditional default probability of obligor j is given by

$$p_j = Pr\{Y_j < \alpha_j\} = \Phi(\alpha_j) \tag{4}$$

where Φ denotes the normal cumulative density function. For simplicity, we have dropped the dependence on time, t, from the notation.Thus, the unconditional threshold, α_j, is obtained by the inverse of Equation (4):

$$\alpha_j = \Phi^{-1}(p_j) \tag{5}$$

The conditional probability of default is then the probability that the credit worthiness index falls below the threshold in a given scenario:

$$p_j(\boldsymbol{Z}) = Pr\{Y_j < \alpha_j | \boldsymbol{Z}\} \tag{6}$$

$$= Pr\left\{\sum_{k=1}^{q^c} \beta_{jk} Z_k + \sigma_j \varepsilon_j < \alpha_j \mid \boldsymbol{Z}\right\}$$

$$= Pr\left\{\varepsilon_j < \frac{\alpha_j - \sum_{k=1}^{q^c} \beta_{jk} Z_k}{\sigma_j}\right\}$$

$$= \Phi\left(\frac{\alpha_j - \sum_{k=1}^{q^c} \beta_{jk} Z_k}{\sigma_j}\right)$$

$$= \Phi(\hat{\alpha}_j(\boldsymbol{Z}))$$

The **conditional threshold**, $\hat{\alpha}_j(\boldsymbol{Z})$, is the threshold that the idiosyncratic component of obligor *j*, ε_j, must fall below for default to occur in scenario $\boldsymbol{Z}$.

Note that obligor credit worthiness index correlations are uniquely determined by the default model and the multi-factor model, which links the index to the credit driver returns. The correlations between obligor defaults are then obtained from the functional relationship between the index and the event of default, as determined by the Merton model. For example, the indices of obligors that belong to the same sector are perfectly correlated if their idiosyncratic component is zero.

Part 3. Obligor exposures and recoveries in a scenario

Define the exposure to an obligor *j* at time *t*, V_j, as the amount that will be lost due to outstanding transactions with that obligor if default occurs, unadjusted for future recoveries. An important property of PCR_SD is the assumption that obligor exposure is deterministic, not scenario dependent: $V_j \neq f(\boldsymbol{Z})$.

The economic loss if obligor *j* defaults in any scenario is

$$L_j(\boldsymbol{Z}) = V_j \cdot (1 - \gamma_j) \tag{7}$$

where γ_j is the recovery rate, expressed as a fraction of the obligor exposure. Recovery, in the event of default, is also assumed to be deterministic. (Expressing the recovery amount as a fraction of the exposure value at default does not necessarily imply instantaneous recovery of a fraction of the exposure when default occurs.)

The distribution of conditional losses for each obligor is given by

$$L_j(\mathbf{Z}) = \begin{cases} V_j \cdot (1-\gamma_j) & \text{with prob.} \quad p_j(\mathbf{Z}) \\ 0 & \text{with prob.} \quad 1-p_j(\mathbf{Z}) \end{cases} \qquad (8)$$

given by sum of the expected losses of each obligor:

$$EL(\mathbf{Z}) = \sum_{j=1}^{N} E\{L_j(\mathbf{Z})\} = \sum_{j=1}^{N} V_j \cdot (1-\gamma_j) \cdot p_j(\mathbf{Z}) \qquad (9)$$

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