Sphere Packings and Foams

Thomas C. Hales

October 2, 2006

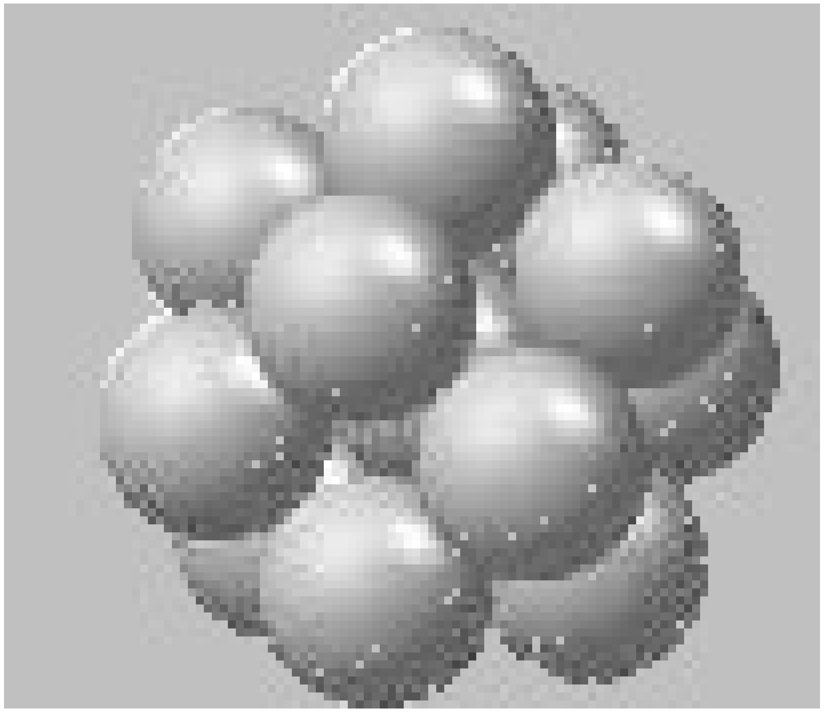
Sphere Packings and Foams

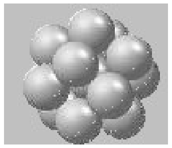
Thomas C. Hales

October 2, 2006

Theorem [Newon-Gregory Problem] It is impossible to arrange thirteen nonoverlapping congruent balls so that they all touch a given (fourteenth) ball.

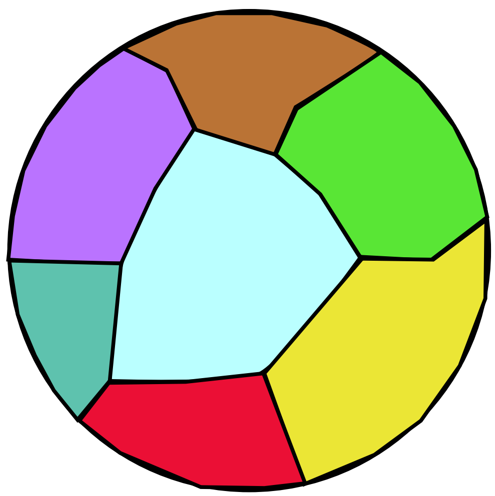
There have been many false proofs of this result!



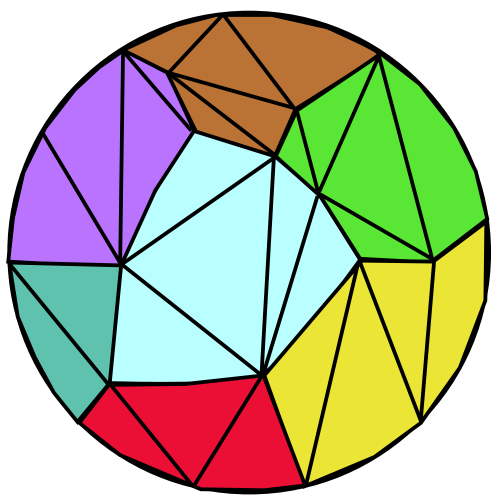


- The false proofs began to appear in the nineteenth century.
- Credit for the first correct proof is due to Shütte and van der Waerden.
- Leech gave a nice proof in the 50s.
- Musin gave another nice proof two years ago.

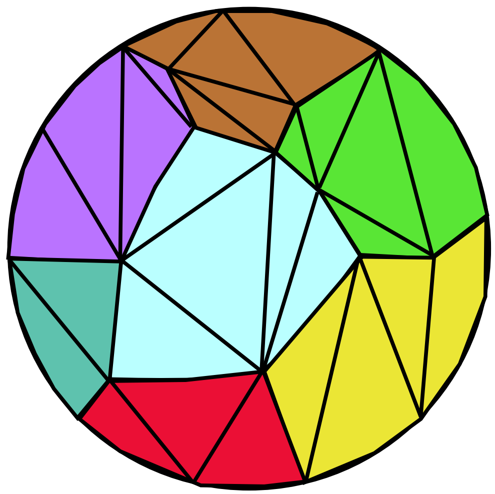
- Proof by contradiction: assume that there are 13 balls.
- Each sphere center projects to a point on the unit sphere.
- Use those points as vertices of a graph on the sphere.
- Add the areas of the polygons, and check that their total area is greater than 4π.



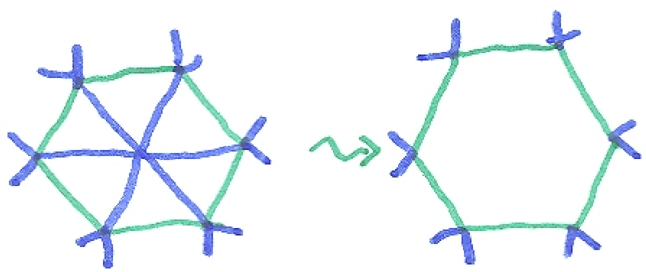
- How did Leech calculate the areas?
- Leech doesn't say. (It probably involved the use of a slide rule.)
- (The reconstruction of the area-calculation in *Proofs from the Book, first edition* is incorrect.)
- The easiest way seems to be by triangulating the sphere, then calculating the areas of triangles.
- In fact, the proof becomes more elementary if we just start with the triangulation!



Leech's proof is really a computer proof in which the computer output has been hidden from view.



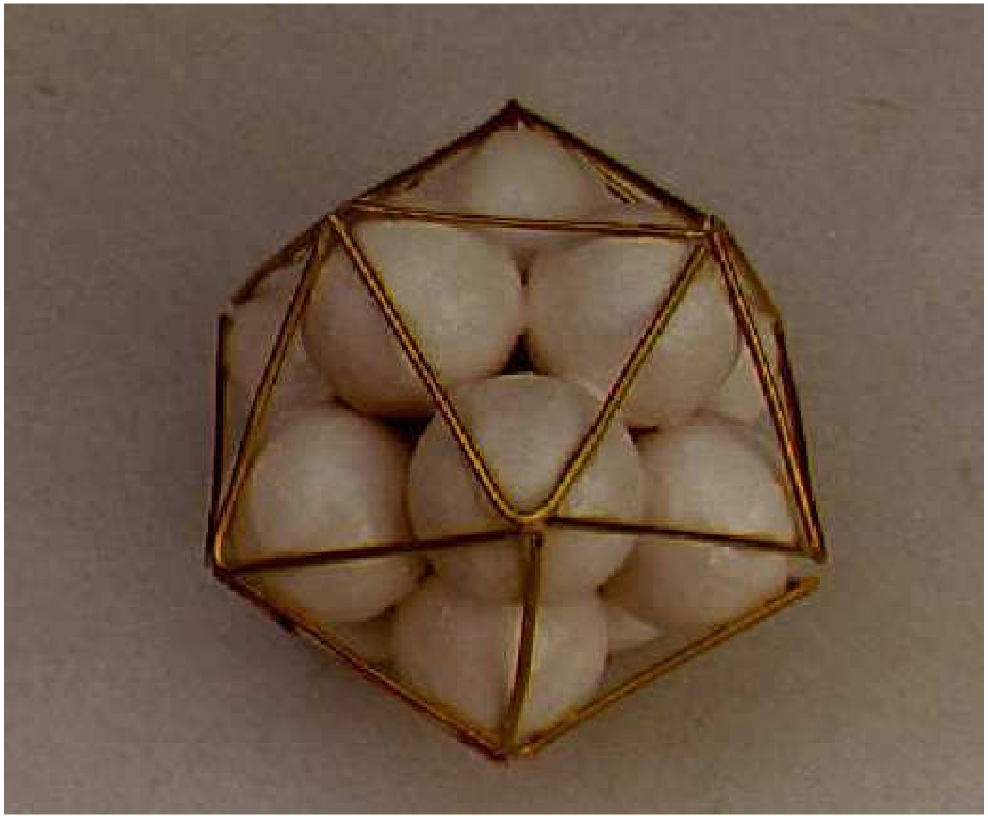
Leech's proof also has a combinatorial side. There is one triangulation that cannot be easily eliminated by adding up the areas. It is a triangulation of the sphere with 12 vertices of degree 5 and one vertex of degree 6. There are 22 triangles.

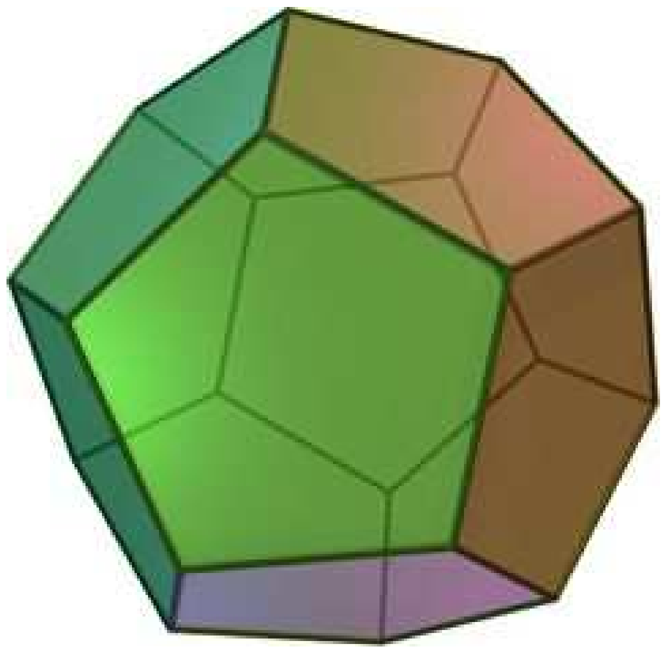


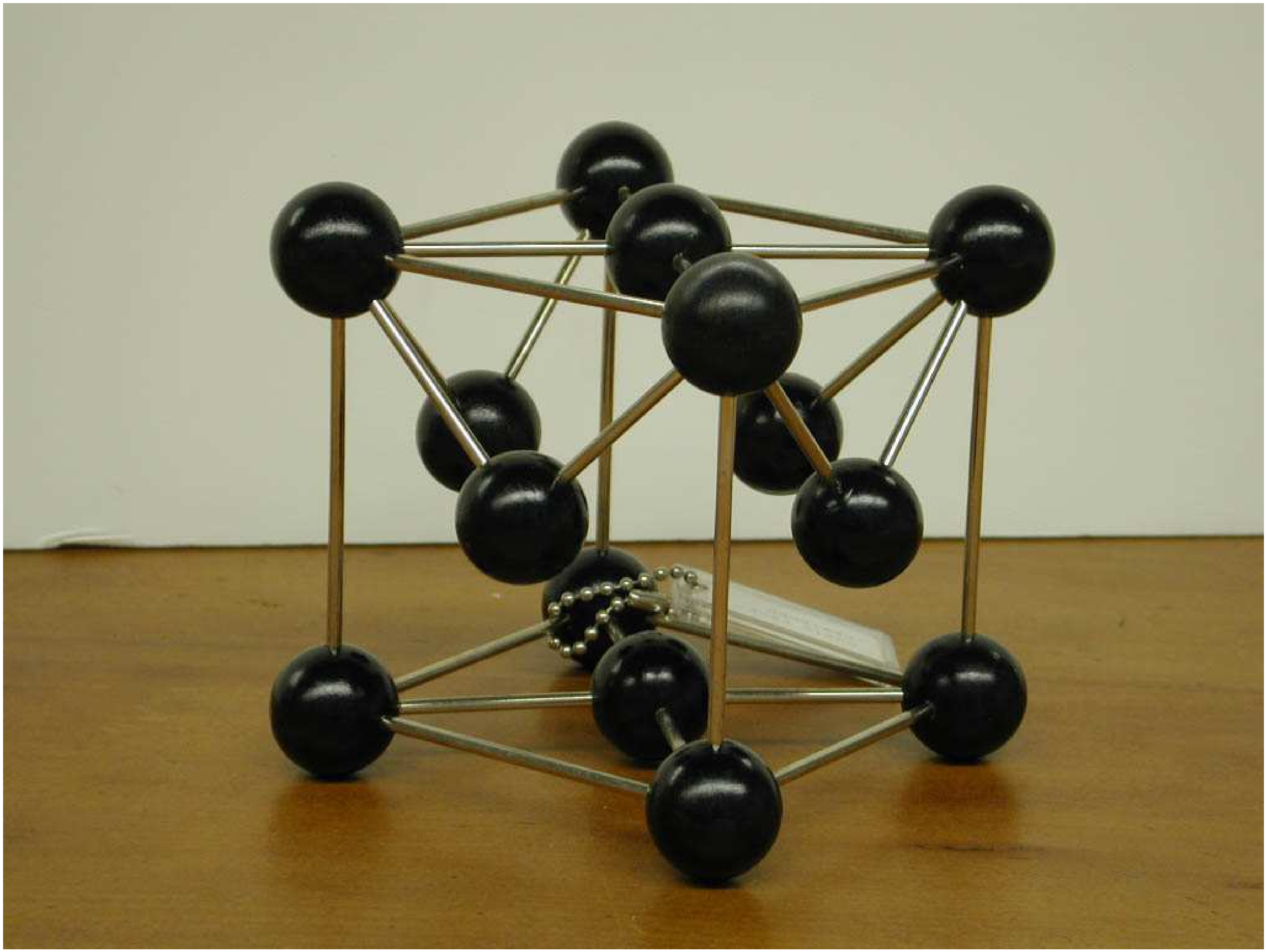
By eliminating the vertex of degree 6 we get a triangulation of a hexagon, with internal vertices of degree 5 and external vertices of degree 4.

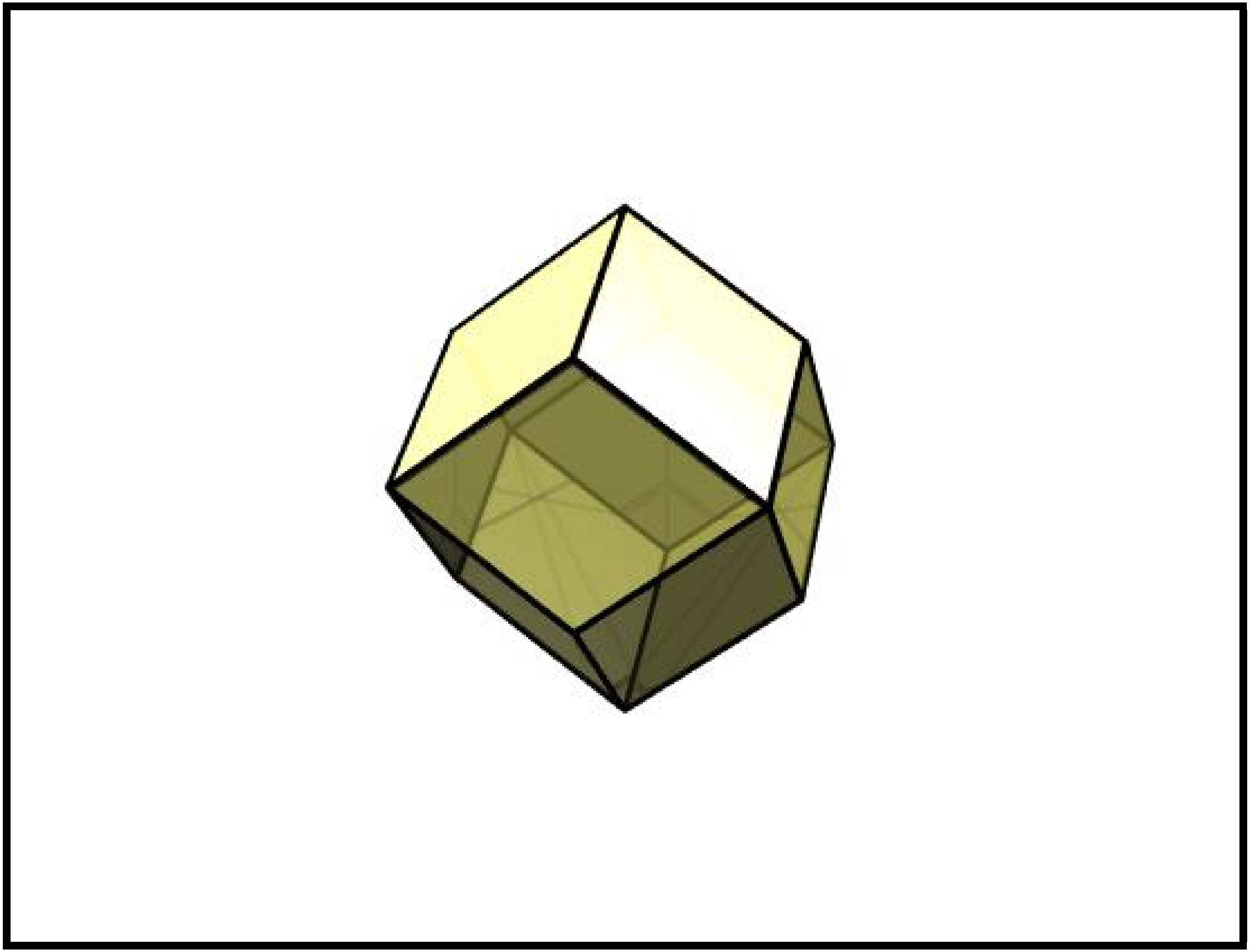
1 Dodecahedral Conjecture

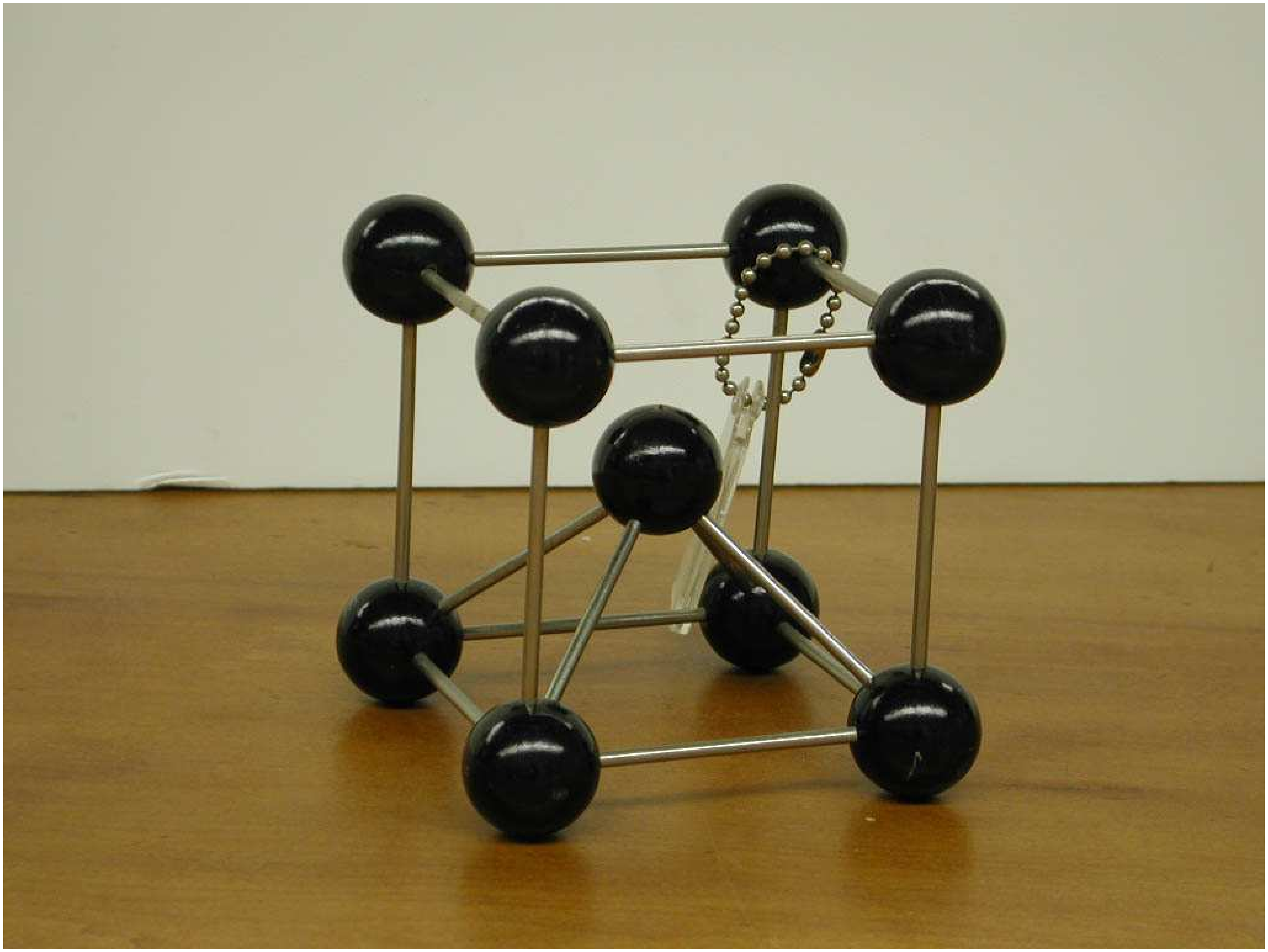
Take any packing of congruent balls in space. The *Voronoi cell* around a ball is the set of points closer to the center of that ball than to the center of any other ball.



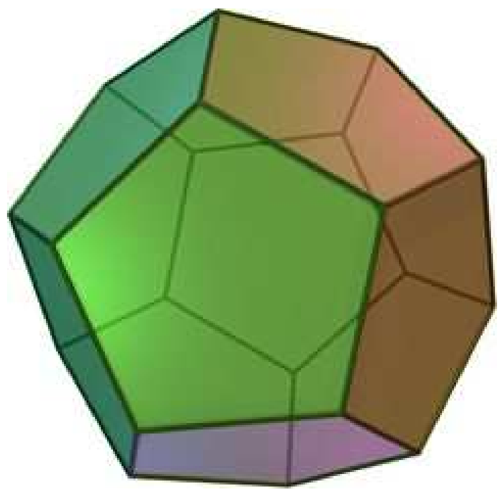






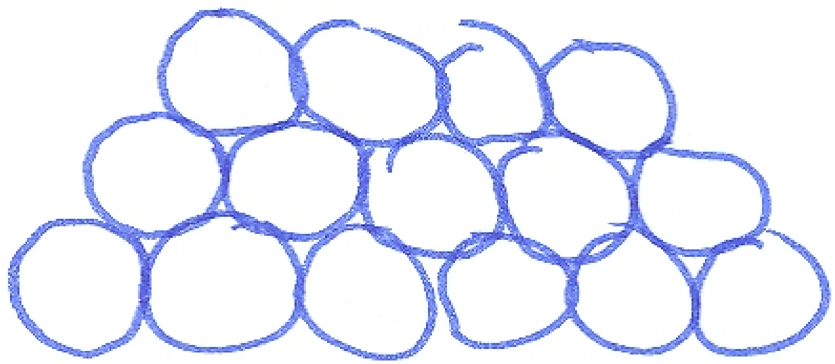


The dodecahedral conjecture asserts that the Voronoi cell of minimum volume is a regular dodecahedron.



Theorem [McLaughlin, 1998] the dodecahedral conjecture is true.

The two dimensional proof is easier to describe.



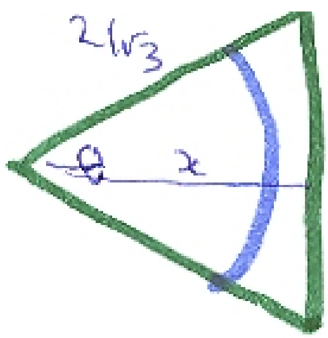
In 2D, the optimal Voronoi cell is a regular hexagon. For the proof of optimality, it is enough to show that each Voronoi cell has density at most $\pi/\sqrt{12}$.

Intersect each Voronoi cell with a disk of radius $\rho = 2/\sqrt{3}$ (the radius of the hexagon).

The truncated Voronoi cell is made up of orange truncated regions and untruncated green regions.

Each truncated orange region has density $1/\rho^2 = 3/4$.

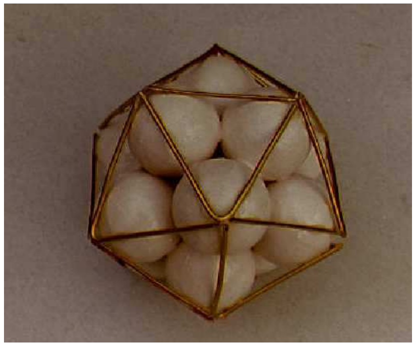
In a green region, the density $\delta(\theta)$ depends on the angle θ of the triangle. (The has two sides of length ρ.) The derivative of δ is not zero. The maximum density is $\pi/\sqrt{12}$.

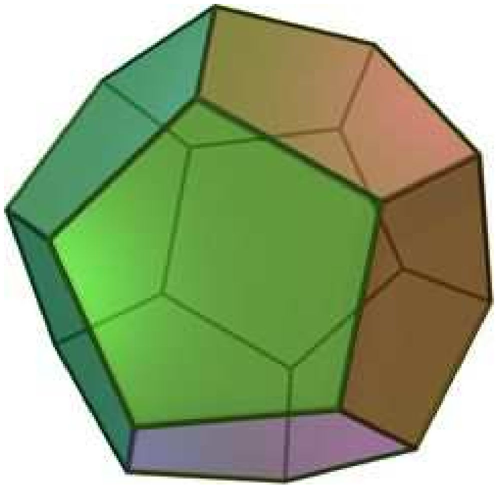


McLaughlin’s proof of the dodecahedral conjecture borrows strategy from this 2D proof and from Leech’s proof of the problem of 13 spheres.

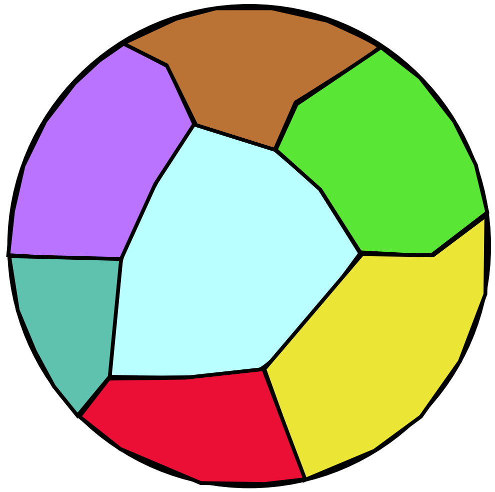
- It is enough to estimate the lower bound of truncated Voronoi cells.
- Make a graph on the surface of the sphere in the style of Leech.
- a vertex for each face of the truncated Voronoi cell.
- an edge if the truncated Voronoi faces meet.

The graph is a sort of dual graph to network of edges and vertices of the Voronoi cell. For example, the graph for the regular dodecahedron is the icosahedron.





Schematically, we get something similar to the picture for Leech.



There are three tasks

- Classify all graphs that are potential counterexamples to the dodecahedral conjecture.
- Give lower bounds for the chunk of volume over a region of the spherical graph.
- Add up the chunks of volumes to see that they always give at least the dodecahedron.

2 Some Open Problems

- The Kissing Number Problem (Newton-Gregory) problem has only been solved in dimensions 2 (6 disks), 3 (12 spheres), 4 (24 spheres), 8 and 24.
- The Voronoi cell minimization problem (Dodecahedral conjecture) has only been solved in dimensions 2 and 3.

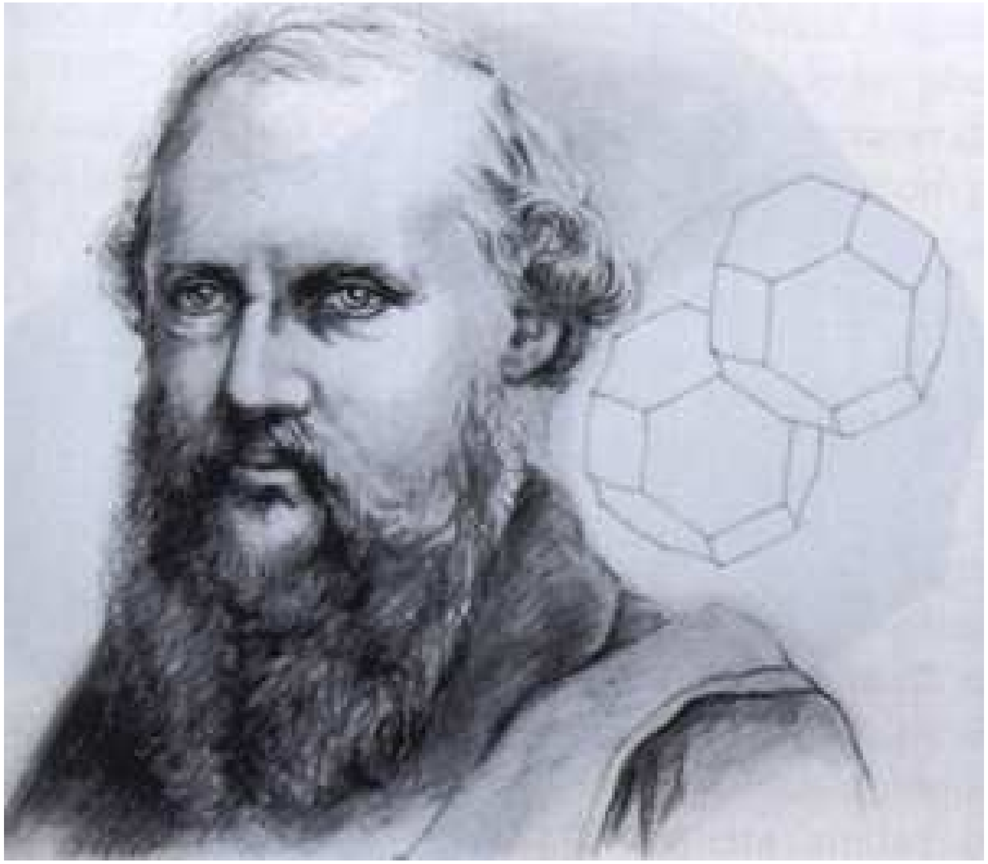
3 Optimal Foams

The isoperimetric inequality states the surface area minimizing way to enclose a region of volume 1 is the sphere.

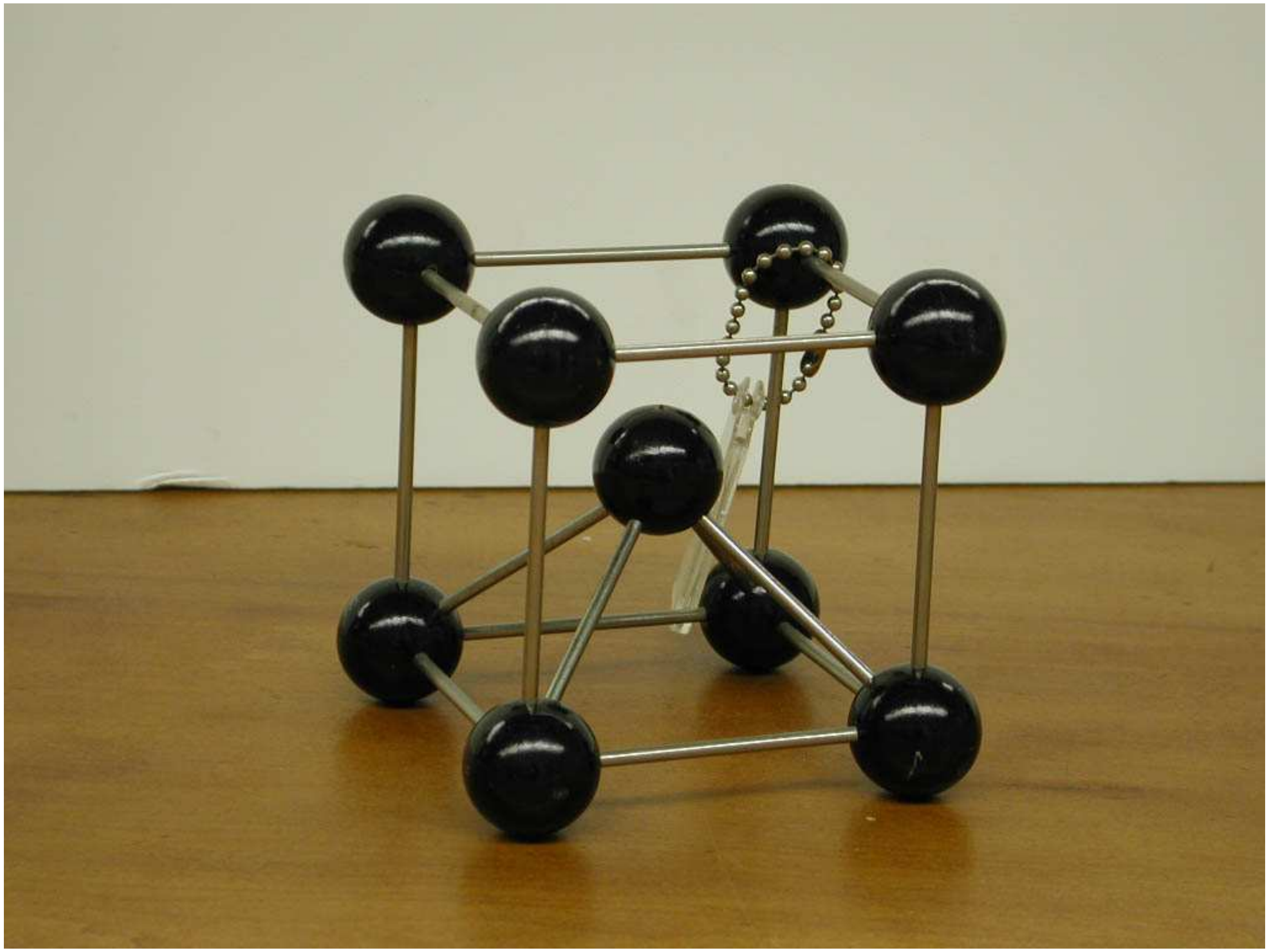
A similar question can be asked with two regions: what is the surface area minimizing way to separately enclose two regions of given volume. The answer (due to Hutchings, Morgan, Ritoré, and Ros) is the double bubble.

The proof has to rule out various alternatives, such as one bubble wrapped around another.

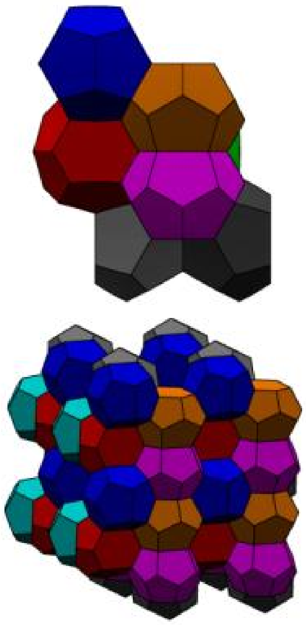
The Kelvin problem (still open) asks for the surface area minimizing way to separate and enclose infinitely many regions of the same volume. Here is Kelvin's proposal.

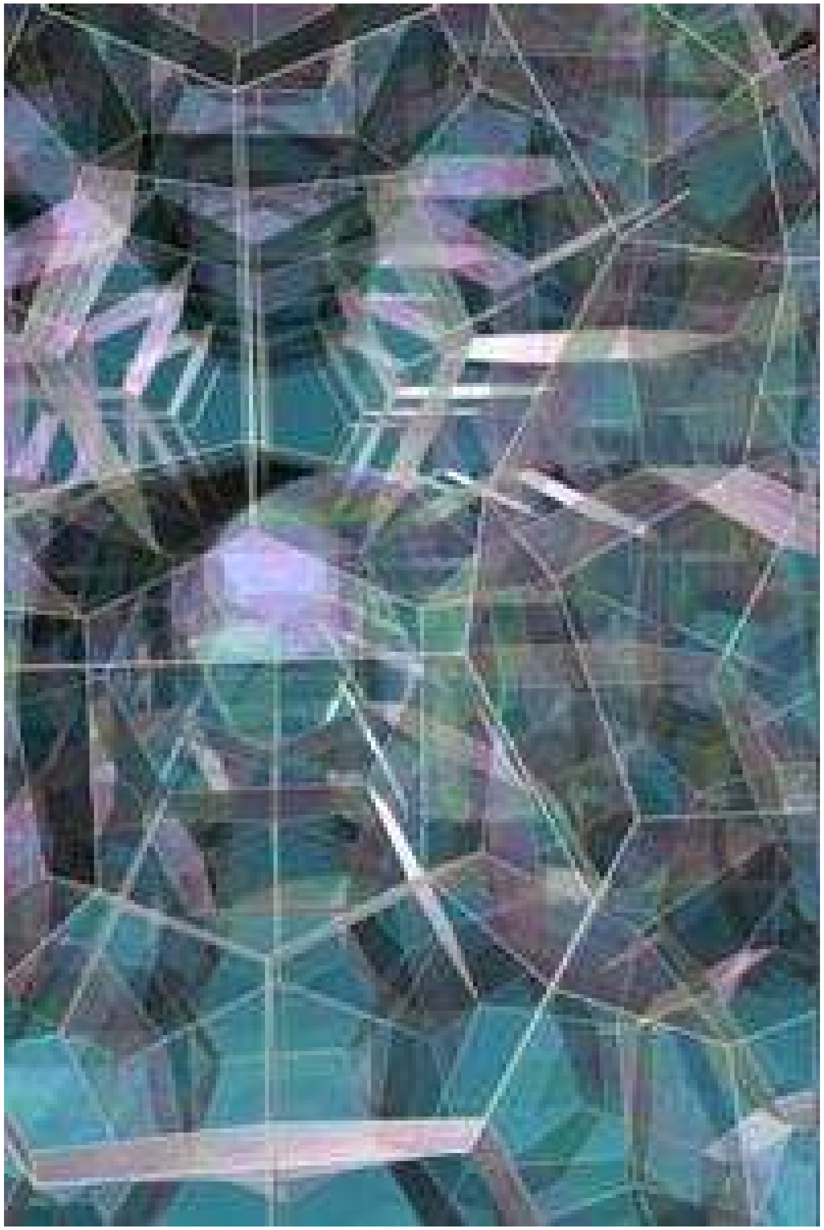


The Kelvin cell is a slightly warped Voronoi cell for the body-centered cubic sphere packing.

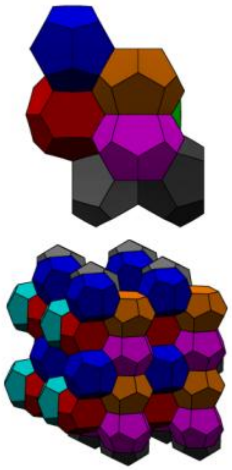


Kelvin’s solution was widely accepted until a counterexample was found by two physicists (Phelan and Weaire) in 1994.



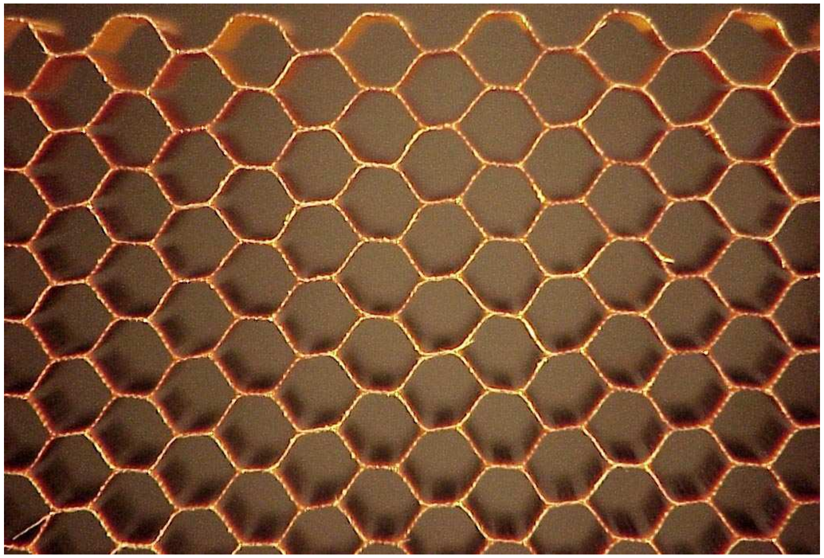


The question of whether the Phelan-Weaire foam is optimal remains an open question.



As I was turning from the subject of packings to foams in my research, Weaire was turning from foams to packings. . .

Theorem (H, 1999) The answer in two dimensions to the foam problem is the regular hexagon. This is the perimeter minimizing partition of the plane into cells of equal area.

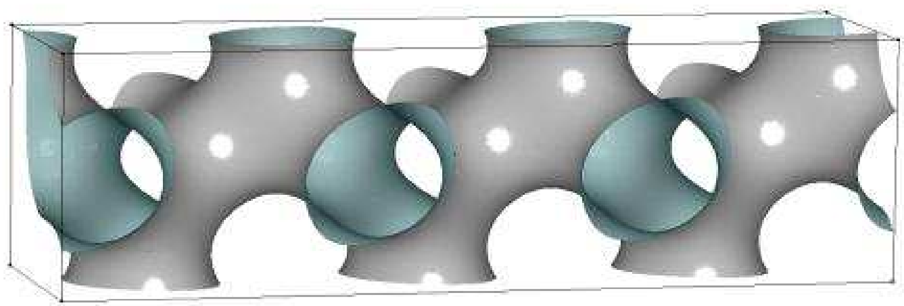


This problem was known to geometers at least 36B.C. and is described by Pappus of Alexandria in his fifth book.

There is an enormous scientific literature on the subject.

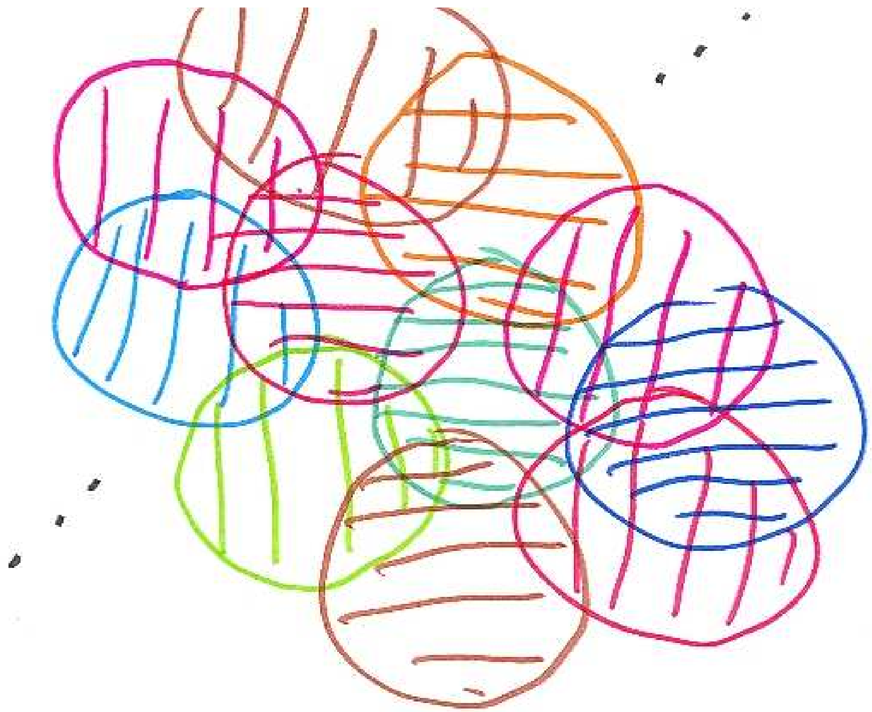
The shape of the honeycomb suggested the rhombic dodecahedron to Kepler.

The foam problem is unsolved in dimensions three and higher. Antonio Ros has proposed a variant of Kelvin's problem, where we partition space into two regions of equal volume in a way that preserves a crystallographic symmetry group.

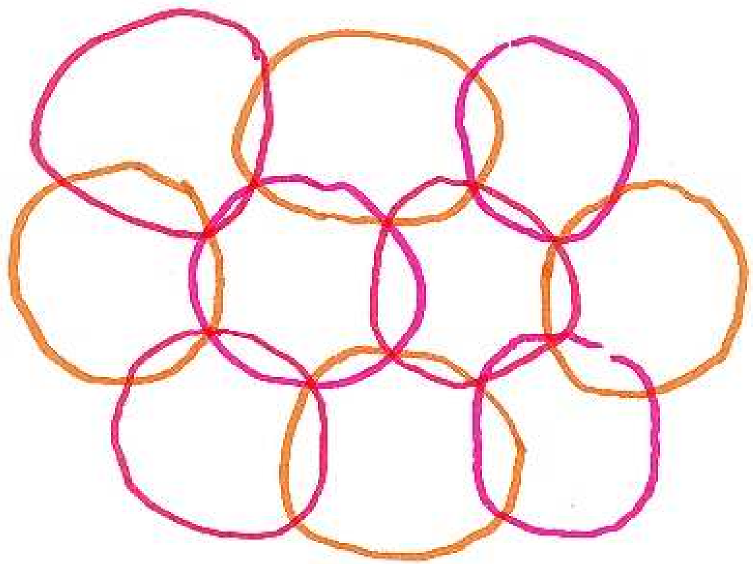


He conjectures this gyroid to be the surface area minimizing partition (within its symmetry class) of space into two equal volumes.

The subject of optimal geometry is full of interesting problems and conjectures. For example, what is the optimal way to cover the plane with congruent disks?



Again, it is the familiar hexagonal arrangement.



In three dimensions and higher, it is completely open. Here is the conjecture in three dimensions: